STRONGLY GENERALIZED NEIGHBORHOOD SYSTEMS

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Abstract. In this paper we define strongly generalized neighborhood systems (in brief strongly GNS) and study their properties. It's proved that every generalized topology μ on X gives a unique strongly GNS $\psi_\mu : X \to \exp(\exp X)$. We prove that if a generalized topology μ is given, then $\mu_{\psi_\mu} = \mu$; and if a strongly GNS ψ is given, then $\psi_{\mu_\psi} = \psi$. Strongly (ψ_1, ψ_2)-continuity is defined. We prove that $f : X \to Y$ is *strongly* (ψ_1, ψ_2)-continuous if and only if it is $(\mu_{\psi_1}, \mu_{\psi_2})$-continuous.

1. Preliminaries

Definitions and theorems mentioned in this section can be found in [1, 2, 3, 4, 5], for more details one can consult them.

Let X be a nonempty set. A collection μ of subsets of X is called a generalized topology on X and the pair (X, μ) is called a generalized topological space, if μ satisfies the following two conditions:

(1) $\phi \in \mu$.
(2) Any union of elements of μ belongs to μ.

Let $\beta \subset \exp(X)$ and $\phi \in \beta$. Then β is called a base for μ if $\mu = \{\cup\beta'; \beta' \subset \beta\}$. We also say μ is generated by β. A generalized topological space (X, μ) is said to be strong if $X \in \mu$. A subset B of X is called μ-open (resp. μ-closed) if $B \in \mu$ (resp. if $X - B \in \mu$). For every $B \subset X$, let $I(B)$ be defined as the largest μ-open subset of X contained in B. Equivalently, $I(B)$ is the union of all μ-open subsets of X contained in B. Let $C(B)$ be defined as the smallest μ-closed subset of X contains B. Equivalently, $C(B)$ is the intersection of all μ-closed subsets of X containing B. The set of all μ-open sets containing a point $x \in X$ will be denoted by μ_x (i.e. $\mu_x = \{U \in \mu; x \in U\}$).

Definition 1.1. [1] Let $\psi : X \to \exp(\exp X)$ satisfies $x \in V$ for every $V \in \psi(x)$. Then we say $V \in \psi(x)$ is a generalized neighborhood (briefly GN) of $x \in X$ and ψ is a generalized neighborhood system (briefly GNS) on X. We denote by $\Psi(X)$ the collection of all GNS's on X.

Theorem 1.2. [1] *Let ψ be a GNS on X and $G \in \mu$ if and only if $G \subset X$ satisfies:*

$$\text{if } x \in G, \text{ then there exists } V \in \psi(x) \text{ such that } V \subset G.$$

Then μ is a generalized topology for X.

In this case we write $\mu = \mu_\psi$.

Theorem 1.3. [1] *If μ is a generalized topology on X, then there exists a GNS $\psi : X \to \exp(\exp X)$ satisfying $\mu = \mu_\psi$. We can suppose ψ fulfills*

$$V \in \mu \text{ for every } V \in \psi(x),\ x \in X.$$

We can define a GNS $\psi_{\mathfrak{a}} \in \Psi(X)$ for every $\mathfrak{a} \subset \exp X$ (thus $\mathfrak{a}$ need not be a generalized topology) by $V \in \psi_{\mathfrak{a}}(x)$ if and only if $x \in V \in \mathfrak{a}$ [1].

Definition 1.4. [1] Let μ_1 and μ_2 be two generalized topologies for X and Y, respectively. A function $f : X \to Y$ is said to be (μ_1, μ_2)-continuous if and only if $V \in \mu_2$ implies $f^{-1}(V) \in \mu_1$.

Definition 1.5. [1] Let ψ_1 and ψ_2 be two GNS's on X and Y, respectively. A function $f : X \to Y$ is said to be (ψ_1, ψ_2)-continuous if and only if for every $x \in X$ and $V \in \psi_2(f(x))$ there exists $U \in \psi_1(x)$ such that $f(U) \subset V$.

Proposition 1.6. [1] *Every (ψ_1, ψ_2)-continuous function is (μ_1, μ_2)-continuous. The converse is not necessarily true.*

The following are about neighborhood systems in topological spaces.

Definition 1.7. [5] A function $\mathfrak{u} : X \to \exp(\exp X)$ is called a *neighborhood system* on X if it satisfies the following four conditions:

(1) If $U \in \mathfrak{u}(x)$, then $x \in U$.
(2) If $U, V \in \mathfrak{u}(x)$, then $U \cap V \in \mathfrak{u}(x)$.
(3) If $U \in \mathfrak{u}(x)$ and $U \subset V$, then $V \in \mathfrak{u}(x)$.
(4) If $U \in \mathfrak{u}(x)$, then there exists $V \in \mathfrak{u}(x)$ such that $V \subset U$ and $V \in \mathfrak{u}(y)$ for every $y \in V$.

Theorem 1.8. [5] *Let $\mathfrak{u} : X \to \exp(\exp X)$ be a neighborhood system on X. Then the set τ of all subsets U of X such that $U \in \mathfrak{u}(x)$ for every $x \in U$ is a topology for X and $\mathfrak{u}(x)$ is precisely the neighborhood system of x relative to the topology τ.*

2. Strongly Generalized Neighborhood Systems

Generalized neighborhood systems (GNS's) are defined and studied in [1, 3] as a generalization of neighborhood systems in topological spaces. But neighborhood systems can be used to redefine topological spaces, and this is not true in GT's, i.e. GT's can't be redefined using GNS's although every GT gives a GNS and vice versa but the two notations are still different;

the following example shows that continuity in GT's is a different notation than continuity in GNS's.

Example 2.1. *Let $X = \{1,2,3\}$ and define $\psi_1, \psi_2 : X \to \exp(\exp X)$ as follows:*

$$\psi_1(1) = \{\{1,2\}\}$$
$$\psi_1(2) = \{\{1,2\}\}$$
$$\psi_1(3) = \{\{2,3\}\},$$

and

$$\psi_2(1) = \{\{1,2\}\}$$
$$\psi_2(2) = \{\{1,2\},\{2,3\}\}$$
$$\psi_2(3) = \{\{1,3\}\}.$$

One can easily show that ψ_1 and ψ_2 are GNS's on X, and $\mu_{\psi_1} = \mu_{\psi_2} = \{\phi, X, \{1,2\}\}$, so there exists at least one $(\mu_{\psi_1}, \mu_{\psi_2})$-continuous bijection from X onto X (consider the identity function on X). We will show there exists no bijection $f : X \to X$ which is (ψ_1, ψ_2)-continuous. We can represent the set of all bijections from X onto X by the symmetric group $\mathbb{S}_3 = \{(1),(12),(13),(23),(123),(132)\}$. One easily can show the following: (1) and (13) are not (ψ_1,ψ_2)-continuous at 2, (23), (123) and (132) are not $(\psi_1,\psi_2)-$continuous at 3, and (12) is not $(\psi_1,\psi_2)-$continuous at 1.

The following is a strengthening of GNH's.

Definition 2.2. Let X be a nonempty set. A function $\psi : X \to \exp(\exp X)$ is said to be a strongly generalized neighborhood system (in brief strongly GNS) if and only if ψ satisfies the following three conditions:

(1) $x \in U$ for every $x \in X$ and every $U \in \psi(x)$.
(2) For every $x \in X$ and every $V \in \psi(x)$, there exists $U \in \psi(x)$ with $U \subset V$ and if $y \in U$, then $U \in \psi(y)$.
(3) For every $U \in \psi(x)$ and every $V \subset X$ if $U \subset V$, then $V \in \psi(x)$.

The collection of all strongly generalized neighborhood systems will be denoted by $\Psi^*(X)$.

In this note we will show that strongly GNS's is a generalization of neighborhood systems in general topology and can be used to redefine GT's.

It is clear that every neighborhood system is a strongly GNS and every strongly GNS is a GNS. One can easily show that ψ_1 in Example 2.1 is a GNS but not a strongly GNS. The following example is a strongly GNS but not a neighborhood system.

Example 2.3. *Let $X = \{1, 2, 3\}$ and define $\psi : X \to \exp(\exp X)$ by*

$$\psi(1) = \{\{1, 2\}, \{1, 3\}, X\}$$
$$\psi(2) = \{\{1, 2\}, \{2, 3\}, X\}$$
$$\psi(3) = \{\{1, 3\}, \{2, 3\}, X\}.$$

It is clear that ψ is a strongly GNS. But since $\{1, 2\}, \{1, 3\} \in \psi(1)$ and $\{1, 2\} \cap \{1, 3\} = \{1\} \notin \psi(1)$, ψ is not a neighborhood system.

Every strongly GNS gives a generalized topology, more precisely we have the following.

Theorem 2.4. *Let $\psi : X \to \exp(\exp X)$ be a strongly GNS on X, and let μ_ψ be the collection of all subsets U of X such that $U \in \psi(x)$ for every $x \in U$. Then μ_ψ is a generalized topology for X.*

Proof. Since there exists no $x \in \phi$, $\phi \in \mu_\psi$. Suppose that $U_\alpha \in \mu_\psi$ for every $\alpha \in \Delta$ and let $x \in \bigcup_{\alpha \in \Delta} U_\alpha$. Then there exists $\alpha_\circ \in \Delta$ such that $x \in U_{\alpha_\circ} \in \mu_\psi$, so that $U_{\alpha_\circ} \in \psi(x)$. Since $U_{\alpha_\circ} \subset (\bigcup_{\alpha \in \Delta} U_\alpha)$, $(\bigcup_{\alpha \in \Delta} U_\alpha) \in \psi(x)$. $\square$

The topology μ_ψ will be called the generalized topology generated on X by the strongly GNS ψ.

The following is a characterization of μ_ψ.

Theorem 2.5. *Let $\psi : X \to \exp(\exp X)$ be a strongly GNS on X, and let μ_ψ be the generalized topology generated on X by the strongly GNS ψ. Then $U \in \mu_\psi$ if and only if for every $x \in U$ there exists $V \in \psi(x)$ such that $x \in V \subset U$.*

Proof. Suppose $U \in \mu_\psi$, and let $x \in U$. It's clear that if we take $V = U$, then $V \in \psi(x)$ and $x \in V \subset U$. Conversely; suppose that for every $x \in U$ there exists $V \in \psi(x)$ such that $x \in V \subset U$. For every $x \in U$ let $V_x \in \psi(x)$ such that $x \in V_x \subset U$. Since $x \in V_x \subset U$, we have $U \in \psi(x)$ for every $x \in U$. That is $U \in \mu_\psi$. $\square$

We proved that every Strongly GNS ψ gives a generalized topology μ_ψ. The next theorem shows the converse.

Theorem 2.6. *Let μ be a generalized topology on X and define $\psi_\mu : X \to \exp(\exp X)$ as follows:*

$A \in \psi_\mu(x)$ if and only if there exists $U \in \mu$ with $x \in U$ and $U \subset A$.

Then ψ_μ is a strongly GNS on X.

Proof. We shall show that ψ_μ satisfies the three conditions of strongly GNS's mentioned in Definition 2.2.

(1) It is clear that if $A \in \psi_\mu(x)$, then $x \in A$.

(2) Let $x \in X$ and $U \in \psi_\mu(x)$. Then there exists $V \in \mu$ such that $x \in V$ and $V \subset U$. Since $V \in \mu$, we have $V \in \psi_\mu(y)$ for every $y \in V$. Also, it is clear that $V \in \psi_\mu(x)$.
(3) Let $U \in \psi_\mu(x)$ and $U \subset V$. Since $U \in \psi_\mu(x)$, there exists $M \in \mu$ such that $x \in M \subset U \subset V$, so that $V \in \psi_\mu(x)$.

Hence, ψ_μ is a strongly GNS on X. □

Theorem 2.7. *Let μ be a generalized topology for X, $\psi_\mu : X \to \exp(\exp X)$ be the strongly GNS generated on X by μ, and μ_{ψ_μ} be the generalized topology for X generated by ψ_μ. Then $\mu_{\psi_\mu} = \mu$.*

Proof. First, we will show that $\mu \subset \mu_{\psi_\mu}$. Let $U \in \mu$. Then $U \in \psi_\mu(x)$ for every $x \in U$, which means $U \in \mu_{\psi_\mu}$. Conversely, let $U \in \mu_{\psi_\mu}$ and $x \in U$. Then, $U \in \psi_\mu(x)$. So there exists $V_x \in \mu$ such that $x \in V_x \subset U$ which implies $U = \bigcup_{x \in U} V_x$. Since $V_x \in \mu$ for every $x \in U$, $U \in \mu$. □

Theorem 2.8. *Let $\psi : X \to \exp(\exp X)$ be a strongly GNS on X, μ_ψ be the generalized topology generated on X by ψ, and $\psi_{\mu_\psi} : X \to \exp(\exp X)$ be the strongly GNS on X generated by μ_ψ. Then $\psi_{\mu_\psi} = \psi$.*

Proof. It suffices to show that $\psi_{\mu_\psi}(x) = \psi(x)$ for every $x \in X$. Let $U \in \psi(x)$. Then there exists $V \in \psi(x)$ such that $x \in V \subset U$ and $V \in \psi(y)$ for every $y \in V$, so that $V \in \mu_\psi$. Since $x \in V \in \mu_\psi$, $V \in \psi_{\mu_\psi}(x)$. But $V \subset U$, so $U \in \psi_{\mu_\psi}(x)$. Conversely, let $U \in \psi_{\mu_\psi}(x)$. Then there exists $V \in \mu_\psi$ such that $x \in V$ and $V \subset U$, which means $V \in \psi(x)$. Since $V \subset U$ and $V \in \psi(x)$, $U \in \psi(x)$. □

Theorem 2.9. *Let $\psi_1, \psi_2 : X \to \exp(\exp X)$ be two strongly GNS's on X with $\mu_{\psi_1} = \mu_{\psi_2}$. Then $\psi_1 = \psi_2$.*

Proof. It suffices to show that $\psi_1(x) = \psi_2(x)$ for every $x \in X$. Let $U \in \psi_1(x)$. Then there exists $V \in \psi_1(x)$ such that $V \subset U$ and for every $y \in V$, $V \in \psi_1(y)$. So $V \in \mu_{\psi_1}$. Since $\mu_{\psi_1} = \mu_{\psi_2}$, $V \in \mu_{\psi_2}$. So that $V \in \psi_2(y)$ for every $y \in V$. But $x \in V$, so $V \in \psi_2(x)$. Since $V \subset U$ and $V \in \psi_2(x)$, $U \in \psi_2(x)$. Which implies $\psi_1(x) \subset \psi_2(x)$. Using the same argument we have $\psi_2(x) \subset \psi_1(x)$. Hence, $\psi_1(x) = \psi_2(x)$ for every $x \in X$ and $\psi_1 = \psi_2$. □

Corollary 2.10. *For any generalized topology μ, ψ_μ is unique.*

Proof. Let ψ_1 and ψ_2 be two strongly GNS on X generated by μ. Then $\mu_{\psi_1} = \mu = \mu_{\psi_2}$. By the previous theorem we have $\psi_1 = \psi_2$. □

Theorem 2.11. *Let $\mathfrak{B}$ be a collection of subsets of X and $\psi : X \to \exp(\exp X)$ be a strongly GNS satisfying $A \in \psi(x)$ if and only if there exists $U \in \mathfrak{B}$ such that $x \in U \subset A$. Then $\mathfrak{B}$ is a $\mu-$base for μ_ψ.*

Proof. Let $\mu(\mathfrak{B})$ be the generalized topology generated on X by $\mathfrak{B}$. We shall show $\mu(\mathfrak{B}) = \mu_\psi$. Let $U \in \mu_\psi$ and $x \in U$. Then $U \in \psi(x)$, so there exists $U_x \in \mathfrak{B}$ such that $x \in U_x \subset U$; hence, $U = \bigcup_{x \in U} U_x$. Since $U_x \in \mathfrak{B}$ for every $x \in U$, $U \in \mu(\mathfrak{B})$, which implies $\mu_\psi \subset \mu(\mathfrak{B})$. Conversely, let $U \in \mu(\mathfrak{B})$. Then $U = \bigcup_{\alpha \in \Delta} U_\alpha$ such that $U_\alpha \in \mathfrak{B}$ for every $\alpha \in \Delta$. If $x \in U$, then $x \in U_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. Since $x \in U_{\alpha_\circ} \in \mathfrak{B}$ and $U_{\alpha_\circ} \subset U$, $U \in \psi(x)$, i.e. $U \in \psi(x)$ for every $x \in U$, which implies $U \in \mu_\psi$, so that $\mu(\mathfrak{B}) \subset \mu_\psi$ and $\mu(\mathfrak{B}) = \mu_\psi$. □

Definition 2.12. Let $\psi_1 : X \to \exp(\exp X)$ and $\psi_2 : Y \to \exp(\exp Y)$ be strongly GNS's on X and Y, respectively. A function $f : X \to Y$ is said to be strongly $(\psi_1, \psi_2)-$continuous if and only if for every $x \in X$ and every $U \in \psi_2(f(x))$ there exists $V \in \psi_1(x)$ such that $f(V) \subset U$.

Theorem 2.13. *Let $\psi_1 : X \to \exp(\exp X)$ and $\psi_2 : Y \to \exp(\exp Y)$ be two strongly GNS's on X and Y, respectively. A function $f : X \to Y$ is strongly $(\psi_1, \psi_2)-$continuous if and only if f is $(\mu_{\psi_1}, \mu_{\psi_2})-$continuous.*

Proof. $\to$) Suppose that f is strongly $(\psi_1, \psi_2)-$continuous. Let $U \in \mu_{\psi_2}$, we shall show that $f^{-1}(U) \in \mu_{\psi_1}$. Let $x \in f^{-1}(U)$. Since $U \in \mu_{\psi_2}$ and $f(x) \in U$, $U \in \psi_2(f(x))$. But f is strongly $(\psi_1, \psi_2)-$continuous, so there exists $V \in \psi_1(x)$ such that $f(V) \subset U$. Since $V \in \psi_1(x)$, there exists $H \in \psi_1(x)$ such that $H \subset V$ and $H \in \psi_1(y)$ for every $y \in H$, which implies $H \in \mu_{\psi_1}$. It is clear that $x \in H \subset f^{-1}(f(H)) \subset f^{-1}(f(V)) \subset f^{-1}(U)$, so that for every $x \in f^{-1}(U)$ there exists $H \in \mu_{\psi_1}$ such that $x \in H \subset f^{-1}(U)$, which means $f^{-1}(U) \in \mu_{\psi_1}$ and f is $(\mu_{\psi_1}, \mu_{\psi_2})-$continuous.

($\leftarrow$) Suppose that f is $(\mu_{\psi_1}, \mu_{\psi_2})-$continuous. Let $x \in X$ and $U \in \psi_2(f(x))$. Then there exists $V \in \psi_2(f(x))$ such that $V \subset U$ and $V \in \psi_2(y)$ for every $y \in V$, which means $V \in \mu_{\psi_2}$ and $f(x) \in V$. Since f is $(\mu_{\psi_1}, \mu_{\psi_2})-$continuous and $f(x) \in V$, $x \in f^{-1}(V) \in \mu_{\psi_1}$. So that $f^{-1}(V) \in \psi_1(x)$, but $f(f^{-1}(V)) \subset V \subset U$, hence, f is strongly (ψ_1, ψ_2)-continuous. □

The following corollary with Example 2.1 shows another difference between GNS's and strongly GNS's.

Corollary 2.14. *If $\psi_1, \psi_2 : X \to \exp(\exp X)$ are two strongly GNS's on X with $\mu_{\psi_1} = \mu_{\psi_2}$, then the identity function $I : X \to X$ (defined by $I(x) = x$ for every $x \in X$) is strongly (ψ_1, ψ_2)-continuous.*

Proof. Since $\mu_{\psi_1} = \mu_{\psi_2}$, I is $(\mu_{\psi_1}, \mu_{\psi_2})$-continuous. Using the previous theorem we have I is strongly (ψ_1, ψ_2)-continuous. □

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