SOLVING SUDOKU: STRUCTURES AND STRATEGIES

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Abstract. We intend to solve Sudoku puzzles using various rules based on the structures and properties of the puzzle. In this paper, we shall present several structures related to either one potential solution or two potential solutions.

1. Introduction

A *Sudoku* puzzle is a 9×9 grid that is partially filled with integers from 1 to 9 as *clues*. A puzzle is given in Figure 1. The solution to a puzzle is a fully filled grid with no duplications in each row, column, and each of the nine 3×3 squares. A puzzle is considered valid if it can be solved uniquely. We also want to consider a well-constructed puzzle to be minimal, that is, removal of any clue results in multiple solutions. Provan [3] gives a more general definition of Sudoku. McGuire, Tugemann, and Civario [2] prove the minimum number of clues for a Sudoku is 17. Surprisingly, we find most 17-clue puzzles are easy to solve.

3						7		
	1				8			
		4				1		
2			1			5	9	
	4			5			3	
		8			9			
	8	1	5			2		
	5		7				6	
	6			2				5

Figure 1. π-Sudoku

There are numerous books and articles on Sudoku. We do not find any deterministic algorithms to solve all puzzles. We shall study the structure and properties of Sudoku puzzles and establish some strategies for solving puzzles deterministically, i.e. without trial-and-error.

2. Elementary Strategies

We consider each row of nine cells as a *row block*, each column of nine cells as a *column block*, and each of the nine 3×3 cells as a *square block*. There are nine blocks of each type. Now, we may also consider every cell to be at the intersection of a row block, a column block, and a square block.

We number the nine row blocks from top down as R_1, R_2, …, R_9, the nine column blocks from left to right as C_1, C_2, …, C_9, and the nine square blocks in row-major order as S_1, S_2, …, S_9. We use c_{ij} to denote the cell in row i and column j.

If only one cell in a block is unfilled, then the solution to the cell is the missing ninth number. The following rule is immediate.

Theorem 1. *Let c be an unfilled cell. If every number except N appears in at least one block of c, then the solution to c is N.*

We will also simply state another rule that is often used. It follows from the idea that in each block, one number appears exactly once.

Theorem 2. *Let c be an unfilled cell in a square block S and let N be a number that does not appear in any blocks containing c. If, in S, every unfilled cell other than c lies in a row block or a column block that contains N implicitly or explicitly, then the solution to c is N.*

In Figure 1, the cell c_{83} in the square block S_7 is unfilled. We examine other unfilled cells in S_7. The cells c_{71}, c_{81}, and c_{91} are in column block C_1 which contains 2. And the cell c_{93} lies in row block R_9 containing 2. With Theorem 2, we can conclude that the solution to cell c_{83} is 2.

Applying Theorem 2 several times to the π-Sudoku in Figure 1, we reach the partially solved puzzle in Figure 2.

We obtain the following rule from the Pigeonhole Principle.

Theorem 3. *Let B be a block with n unfilled cells, and let $N_1, N_2, ..., N_m$ be numbers that do not appear in B, where $m < n$. If these m N_i's are possible solutions only to certain m cells in B, then these N_i's are the only possible solutions to these m cells.*

In Figure 2, square block S_1 has five unfilled cells. Two numbers 2 and 9 do not appear in S_1 and can only be solutions to cells c_{12} and c_{32} by Theorem 2. We apply Theorem 3 to conclude that the only possible solutions to these two cells are 2 and 9. From this point, we use Theorem 2 to find that the solution to cell c_{23} is 7.

3						7		
	1				8			
8		4				1		
2			1			5	9	
	4	9		5			3	
5		8			9			
	8	1	5			2		
	5	2	7				6	
	6	3		2				5

FIGURE 2. π-Sudoku: Applications of Theorem 2.

3. Almost 2-Perfect Components

When we are solving a Sudoku puzzle, we often identify possible solutions for an unsolved cell. A *2-perfect component* is a set of unfilled cells in which if a number is a possible solution, then it is a possible solution to exactly two unfilled cells of the set in every block.

	21							21
	21		21					
			21					21

FIGURE 3. A 2-perfect component

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In Figure 3, we assume that 1 and 2 are only possible solutions to these six cells. According to Theorem 3, numbers 1 and 2 are not solutions to any cells in blocks R_1, R_2, R_3, C_3, C_5, C_9, S_1, S_2, and S_3. In order to see the pattern better, we leave all other cells blank. The following theorem shows that no 2-perfect component exists in a Sudoku puzzle.

Theorem 4. *If there is a 2-perfect component, then the puzzle is not solvable.*

Proof. Assume that there exists a solution to a 2-perfect component. Then this solution consists of one of the two numbers which are only possible solutions to these cells. Since this component is a 2-perfect component, the other of the two numbers form another solution. It contradicts the uniqueness of the solution to a Sudoku puzzle. Therefore, in a valid Sudoku puzzle, a 2-perfect component does not exist. □

The above theorem provides a means to solve some puzzles. An *almost 2-perfect component* is a set of unfilled cells in which, among all potential solutions, exactly one of them appears three times in its blocks while all others appear two times only. That is just one additional number as a possible solution to one cell of a 2-perfect component.

The cell containing three potential solutions is called the *pivot cell* and that third number is called the *pivot number*. In an almost 2-perfect component, removal of the pivot number would convert it to a 2-perfect component. Therefore, we have the following two useful corollaries to solve Sudoku puzzles.

Corollary 5 (Three-Number Rule)**.** *The solution to the pivot cell in a almost 2-perfect component is the pivot number.*

Figure 4 provides an example of the application of Corollary 5, the Three-Number Rule, to solve the cell c_{92}.

7	62	32	8	5	9	4	63	1
4	61	31	76	73	2	8	5	9
9	5	8	1	4	63	7	63	2
1	4	7	3	9	8	5	2	6
3	8	5	2	6	4	9	1	7
2	9	6	5	1	7	3	8	4
5	71	9	76	8	61	2	4	3
6	3	4	9	2	5	1	7	8
8	72^1	21	4	73	31	6	9	5

FIGURE 4. The solution to cell c_{92} is 1

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Corollary 6 (Rectangle Rule). *A valid Sudoku puzzle does not contain a 2-perfect component of four cells.*

In the following example, we have solved a puzzle up to this point. We pay attention to five cells with multiple potential solutions. If cell c_{95} is not a 2, then due to the pigeonhole principle, the potential solutions to cells c_{74} and c_{94} must be 6 and 2. Now, these two cells together with cells c_{71} and c_{91} form a 2-perfect component involving four cells. Applying Corollary 6, Rectangle Rule, we conclude that the solution to cell c_{95} is 2.

4	3		1		2	6	8	
5	1	2	8	6	7	3	9	4
8	6							
	5	4	9	3	8		6	
9	2	6		1			3	8
					6	4		9
62			62 753	4		8		
	4	5		8	1	9	2	6
62			62 75	2 975			4	3

Figure 5. Application of Rectangle Rule

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4. Alternate Cycle-of-N

We shall define several path-like structures. First we consider the adjacency between two unfilled cells. If N is a potential solution to $c_{i_1j_1}$ and $c_{i_2j_2}$ that are in the same block, then we say these two cells are *adjacent* and we write $c_{i_1j_1} \leftrightarrow c_{i_2j_2}$. Furthermore, if N is a potential solution only to these two cells in the block, we call them *2-adjacent* and, when it becomes necessary, we write $c_{i_1j_1} \Leftrightarrow c_{i_2j_2}$.

Assume that N is a potential solution to m cells $c_{i_1j_1}, c_{i_2j_2}, \ldots, c_{i_mj_m}$. If cells $c_{i_tj_t}$ and $c_{i_{t+1}j_{t+1}}$ are adjacent for $1 \leq t \leq m-1$, then the m cells form a *walk-of-N* of length $m-1$. We use $c_{i_1j_1} \leftrightarrow c_{i_2j_2} \leftrightarrow \cdots \leftrightarrow c_{i_mj_m}$ to denote this walk. Again, we may use "$\Leftrightarrow$" instead of "$\leftrightarrow$" when it is applicable and necessary. A walk-of-N is *closed* if the "first" cell and the "last" cell are identical. A *path-of-N* is a walk-of-N with no repeated cells. A *cycle-of-N* is a closed walk-of-N in which there are no repeated cells.

Now, let $c_{i_1j_1} \leftrightarrow c_{i_2j_2} \leftrightarrow \cdots \leftrightarrow c_{i_mj_m} \leftrightarrow c_{i_1j_1}$ be an cycle-of-N of length $m \geq 5$. If $c_{i_{2t}j_{2t}} \Leftrightarrow c_{i_{2t+1}j_{2t+1}}$, then we call this cycle-of-N an *alternate* *cycle-of-N*. Namely, every other adjacency is definitely a 2-adjacency starting from the second one: $c_{i_2j_2} \Leftrightarrow c_{i_3j_3}$, $c_{i_4j_4} \Leftrightarrow c_{i_5j_5}, \ldots$. Note if the length m is odd, the first and the last adjacency do not need to be a 2-adjacency. We call $c_{i_1j_1}$ the *pivot* cell. The following theorem provides a tool in solving more difficult Sudoku puzzles.

Theorem 7. *If an alternate cycle-of-N of odd length is formed when solving a Sudoku puzzle, then N is not the solution to the pivot cell.*

Proof. Let $c_{i_1j_1} \leftrightarrow c_{i_2j_2} \Leftrightarrow c_{i_3j_3} \leftrightarrow \cdots \leftrightarrow c_{i_{m-1}j_{m-1}} \Leftrightarrow c_{i_mj_m} \leftrightarrow c_{i_1j_1}$ be an alternate cycle-of-N of length $m = 2k+1$ for some $k \geq 2$. The pivot cell is $c_{i_1j_1}$. Assume that N is the solution to $c_{i_1j_1}$. Then N is not a solution to $c_{i_2j_2}$, which leads to the conclusion that N is a solution to $c_{i_3j_3}$ due to 2-adjacency. Repeating the argument, we find that the number N is the solution to every odd-indexed cell of the cycle: $c_{i_1j_1}, c_{i_3j_3}, \ldots, c_{i_mj_m}$, since these cells lie on an alternate cycle-of-N. However, $c_{i_1j_1}$ and $c_{i_mj_m}$ belong to the same block and they may not have the same solution. The contradiction proves the theorem. $\square$

Now, we go back to the partial solution of π-Sudoku shown in Figure 6. We observe that the potential solutions to the cells c_{78}, c_{75}, c_{15}, c_{19}, and c_{28} include 4. In the figure, we denote this using a 4+.

3	7+	85		4+		6	1	4+
6	1			8		5	4+	7
85	7+	4		6	1	3		
4			1	32		7	5	
7+		7+		5		9	3	1
				7	9	4		
71	8	71	6	4+	5	2	4+	3
2	5	3		1		8	6	
9	4	6		32	8	1	7	5

FIGURE 6. Alternate cycle-of-4 of length five in π-Sudoku

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In Figure 6, we observe an alternate cycle-of-4 of length five,

$$c_{78} \leftrightarrow c_{75} \Leftrightarrow c_{15} \leftrightarrow c_{19} \Leftrightarrow c_{28} \leftrightarrow c_{78}.$$

By Theorem 7, number 4 is not the solution to cell c_{78}. Therefore, 9 is the solution to cell c_{78}.

It is difficult to determine a longer alternate cycle-of-N in a puzzle. However, it is often useful to solve a hard Sudoku puzzle. We provide the following example to demonstrate another application of Theorem 7.

		9			3	5	6	
7	2	3			5	1	4	
5	6	4	1		2+	3	2+	
	4	8	5	3	7		1	6
		6			2+	7	3	5
3	5	7	2+	1	6	2+	8	4
6		5	2+	8	1	4	7	2+
		2			4	6		1
4	7	1			9	8	2+	3

FIGURE 7. Alternate cycle-of-2 of length 7.

Figure 7 is a partial solution of another puzzle. There is an alternate cycle-of-2 of length 7:

$$c_{64} \leftrightarrow c_{56} \Leftrightarrow c_{36} \leftrightarrow c_{38} \Leftrightarrow c_{98} \leftrightarrow c_{79} \Leftrightarrow c_{74} \leftrightarrow c_{64}.$$

We conclude, by Theorem 7, that 2 is not the solution to cell c_{64}. Therefore, 9 is its solution.

References

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