COMPACT WEIGHTED COMPOSITION OPERATORS BETWEEN GENERALIZED FOCK SPACES

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Abstract. Let ψ be an entire self-map of the n-dimensional Euclidean complex space $\mathbb{C}^n$ and u be an entire function on $\mathbb{C}^n$. A weighted composition operator induced by ψ with weight u is given by $(uC_\psi f)(z) = u(z)f(\psi(z))$, for $z \in \mathbb{C}^n$ and f is the entire function on $\mathbb{C}^n$. In this paper, we study weighted composition operators acting between generalized Fock-types spaces. We characterize the boundedness and compactness of these operators act between $\mathcal{F}^p_\phi(\mathbb{C}^n)$ and $\mathcal{F}^q_\phi(\mathbb{C}^n)$ for $0 < p, q \leq \infty$. Moreover, we give estimates for the Fock-norm of $uC_\psi : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ when $0 < p, q < \infty$, and also when $p = \infty$ and $0 < q < \infty$.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space, and let dv be the usual Lebesgue volume measure on $\mathbb{C}^n$. Throughout this paper, we assume that $\phi \in C^2(\mathbb{C}^n)$ is a given real valued function on $\mathbb{C}^n$ such that

$$mw_0 < dd^c\phi < Mw_0$$

holds uniformly pointwise on $\mathbb{C}^n$ for some positive constants m and M where d is the usual exterior derivative, $d^c = \frac{i}{4}(\overline{\partial} - \partial)$, and $w_0 = dd^c|\cdot|^2$ is the standard Euclidean Kähler form on $\mathbb{C}^n$. Let $\mathcal{H}(\mathbb{C}^n)$ be the space of all entire functions on $\mathbb{C}^n$. For $0 < p < \infty$, the generalized Fock space $\mathcal{F}^p_\phi(\mathbb{C}^n)$ consists of functions $f \in \mathcal{H}(\mathbb{C}^n)$ such that

$$\|f\|^p_{p,\phi} = \int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)} dv(z) < \infty;$$

that is $\mathcal{F}^p_\phi(\mathbb{C}^n) = L^p(\mathbb{C}^n, e^{-p\phi}dv) \cap \mathcal{H}(\mathbb{C}^n)$. Similarly, for $p = \infty$ one can define the generalized Fock-type space $\mathcal{F}^\infty_\phi(\mathbb{C}^n)$ as $\mathcal{F}^\infty_\phi(\mathbb{C}^n) = L^\infty(\mathbb{C}^n, e^{-\phi}dv) \cap \mathcal{H}(\mathbb{C}^n)$; that is $\mathcal{F}^\infty_\phi$ is the space of functions $f \in \mathcal{H}(\mathbb{C}^n)$ such that

$$\|f\|_{\infty,\phi} = \sup_{z \in \mathbb{C}^n} |f(z)|e^{-\phi(z)} < \infty.$$

For $\alpha > 0$ and $0 < p \leq \infty$, if $\phi(z) = \frac{\alpha}{2}|z|^2$, we get the weighted Fock spaces $\mathcal{F}^p_\alpha$. For the overview of the studies on $\mathcal{F}^p_\alpha$ we refer to the monographs [22] and [26]. If $\phi(z) = \frac{|z|^2}{2} - m\log|z|$, where m is a non-negative integer, we get the Fock-Sobolev spaces $\mathcal{F}^{p,m}$, these spaces were first introduced by H. Cho and K. Zhu in [6]. It is well-known that for $1 \leq p \leq \infty$, $\mathcal{F}^p_\phi$ is a Banach space under the norm $\|\cdot\|_{p,\phi}$. Also, for $0 < p < 1$, the space $\mathcal{F}^p_\phi$ is an F-space under $d(f,g) = \|f-g\|^p_{p,\phi}$. Moreover, the Hilbert space $\mathcal{F}^2_\phi$ is a closed subspace of $L^2(\mathbb{C}^n, e^{-2\phi}dv)$ with the Bergman Kernel $K_\phi(\cdot,\cdot)$. The Bergman orthogonal projection $P : L^2(\mathbb{C}^n, e^{-2\phi}dv) \to \mathcal{F}^2_\phi$ is given by integrating against the kernel $K_\phi(w,z) = K_{\phi,z}(w)$; that is

$$Pf(z) = \int_{\mathbb{C}^n} K_\phi(z,w)f(w)e^{-2\phi(w)}dv(w).$$

It is known for $n = 1$ [7] and for $n \geq 2$ [11] that there are positive constants θ and C such that for any z, $w \in \mathbb{C}^n$

$$|K_\phi(z,w)|e^{-\phi(z)}e^{-\phi(w)} \leq Ce^{-\theta|z-w|}. \tag{1}$$

For any $z \in \mathbb{C}^n$ and $r > 0$, we use

$$B(z,r) = \{w \in \mathbb{C}^n : |w - z| < r\}$$

to denote the Euclidean ball centered at z with radius r. Then using Proposition 3.3 of [19], there are positive constants r, C_1, and C_2 such that for each $z \in \mathbb{C}^n$ and each $w \in B(z,r)$ we have

$$|K_\phi(z,w)|e^{-\phi(z)}e^{-\phi(w)} \geq C_1|K_\phi(z,z)|e^{-2\phi(z)} \geq C_2. \tag{2}$$

Using the previous inequalities (1) and (2), it is easy to see that for some positive constants C_3 and C_4 we have

$$C_3 \leq \|K_{\phi,z}e^{-\phi(z)}\|_{p,\phi} \leq C_4. \tag{3}$$

Let μ be a positive Borel measure on $\mathbb{C}^n$, the average of μ on $B(z,r)$ is $\frac{\mu(B(z,r))}{v(B(z,r))}$. Since the Lebesgue volume $v(B(z,r)) = \int_{B(z,r)} dv \simeq r^{2n}$ is a constant over all $z \in \mathbb{C}^n$, we call $\mu(B(z,r))$ an averaging function of μ. For $t > 0$, we define the t-Berezin transform of μ as follows

$$\tilde{\mu}_t(z) = \int_{\mathbb{C}^n} |K_\phi(z,w)|^t e^{-t(\phi(z)+\phi(w))}d\mu(w).$$

Suppose ψ is an entire function mapping $\mathbb{C}^n$ into itself and u is an entire function on $\mathbb{C}^n$. Then the weighted composition operator uC_ψ is defined on the space $\mathcal{H}(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$ by $(uC_\psi f)(z) = u(z)f(\psi(z))$, for all $f \in \mathcal{H}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. The composition operator C_ψ is a weighted composition operator with the weight function u identically equal to 1. It is well-known that the composition operator $C_\psi f = f \circ \psi$ defines a linear operator C_ψ which is bounded on spaces of entire functions on $\mathbb{C}^n$.

During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of uC_ψ in terms of the function-theoretic properties of the induced maps u and ψ. One of the reasons is that it provides a connection between operator theory and complex analysis and doing that helps to gain a deeper understanding of both areas. For the studies on the spaces of analytic functions, we refer to the monographs [8, 12, 13, 20, 27, 28]. Also, see [22] and [26] for the studies on the spaces of entire functions.

Recently, several authors have studied the boundedness and compactness of weighted composition operators on Fock spaces; for example see [1, 2, 3, 5, 18, 21, 23, 24, 25]. The authors of these papers used classical techniques which were used by many authors in Bergman and Hardy spaces, see for example [8, 9, 10, 17, 20, 27, 28]. In this paper we use similar techniques. The purpose of this paper is to extend some previous results to a very wide class of exponentially weighted Fock spaces. In fact we use Carleson measures techniques to characterize the boundedness and compactness of a weighted composition operator uC_ψ acting between Fock spaces $\mathcal{F}^p_\phi(\mathbb{C}^n)$ and $\mathcal{F}^q_\phi(\mathbb{C}^n)$, for $0 < p, q \leq \infty$. In addition, we give estimates for the Fock-norm of $uC_\psi : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ when $0 < p, q < \infty$, and also when $p = \infty$ and $0 < q < \infty$.

2. Preliminaries

The notion of Carleson measure is a crucial tool in investigating the boundedness and compactness of weighted composition operators on spaces of entire functions. In the early 1960s [4], while he was working on the Corona problem, Carleson developed inequalities that relate a behavior of a function $f \in H^p(\mathbb{D})$ in the unit disk with its behavior on the unit circle. The underlying measures, in those inequalities and their generalizations to other spaces, are frequently called Carleson measures. These measures have been extended and found many applications in the study of composition operators in various spaces of functions, for example see [8, 12, 13, 27, 28] for the study of Carleson measures in Hardy and Bergamn spaces, and see [6, 14, 15, 16, 19] for the study of Carleson measures in Fock-type spaces. Now we define the (p, q)-Fock Carleson measure and vanishing (p, q)-Fock Carleson measure as well.

Let $0 < p, q < \infty$ and let μ be a positive Borel measure. We say μ is a (p, q)-Fock Carleson measure if the inclusion map $\imath_\mu : \mathcal{F}^p_\phi(\mathbb{C}^n) \to L^q(\mathbb{C}^n, e^{-q\phi} d\mu)$ is bounded; that is if there exists a constant C such that for all $f \in \mathcal{F}^p_\phi$ the following inequality holds

$$\left(\int_{\mathbb{C}^n} |f(z)|^q e^{-q\phi(z)} d\mu(z)\right)^{p/q} \leq C \int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)} dv(z).$$

Also, we say μ is a vanishing (p,q)-Fock Carleson measure if the inclusion map $\imath_\mu : \mathcal{F}^p_\phi(\mathbb{C}^n) \to L^q(\mathbb{C}^n, e^{-q\phi}d\mu)$ is compact; that is

$$\lim_{n\to\infty} \int_{\mathbb{C}^n} |f_n(z)|^q e^{-q\phi(z)} d\mu(z) = 0$$

for any bounded sequence $\{f_n\} \in \mathcal{F}^p_\phi$ that converges to zero uniformly on compact subsets of $\mathbb{C}^n$.

The following three lemmas are from [14]. They characterize (vanishing) (p,q)-Fock Carleson measures, when $0 < p, q < \infty$, in terms of the t-Berezin transform $\tilde{\mu}_t$ and the averaging function $\mu(B(\cdot, r))$. These lemmas were first obtained by Schuster and Varolin [19] when $p = q \geq 1$.

Lemma 2.1. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent.*

(1) *μ is a (p,q)-Fock Carleson measure;*
(2) *$\tilde{\mu}_t$ is bounded on $\mathbb{C}^n$ for some (or any) $t > 0$;*
(3) *$\mu(B(\cdot,\delta))$ is bounded on $\mathbb{C}^n$ for some (or any) $\delta > 0$.*

Lemma 2.2. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent.*

(1) *μ is a vanishing (p,q)-Fock Carleson measure;*
(2) *$\tilde{\mu}_t(z) \to 0$ as $z \to 0$ for some (or any) $t > 0$;*
(3) *$\mu(B(z,\delta)) \to 0$ as $z \to 0$ for some (or any) $\delta > 0$.*

Lemma 2.3. *Let $0 < q < p < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent.*

(1) *μ is a (p,q)-Fock Carleson measure;*
(2) *μ is a vanishing (p,q)-Fock Carleson measure;*
(3) *$\tilde{\mu}_t \in L^{p/(p-q)}(dv)$ for some (or any) $t > 0$;*
(4) *$\mu(B(\cdot,\delta)) \in L^{p/(p-q)}(dv)$ for some (or any) $\delta > 0$.*

The following lemma characterizes (vanishing) (∞,q)-Fock Carleson measures, when $0 < q < \infty$, in terms of the t-Berezin transform $\tilde{\mu}_t$ and the averaging function $\mu(B(\cdot,r))$. The proof of this lemma is just a simple modification of the proof of Theorem 2.8 [14] with $p = \infty$. So we omit the trivial proof's details.

Lemma 2.4. *Let $0 < q < \infty$, and let $\mu \geq 0$. Then the following statements are equivalent.*

(1) *μ is a (∞,q)-Fock Carleson measure;*
(2) *μ is a vanishing (∞,q)-Fock Carleson measure;*
(3) *$\tilde{\mu}_t \in L^1(dv)$ for some (or any) $t > 0$;*
(4) *$\mu(B(\cdot,\delta)) \in L^1(dv)$ for some (or any) $\delta > 0$.*

3. Boundedness and Compactness

The results of this section concern boundedness and compactness of weighted composition operators mapping $\mathcal{F}^p_\phi(\mathbb{C}^n)$ into $\mathcal{F}^q_\phi(\mathbb{C}^n)$ for $0 < p, q \leq \infty$. Our results will be expressed in terms of the following integral transform that generalizes the Berezin transform. Let ψ be an entire self-map of $\mathbb{C}^n$ and u be an entire function of $\mathbb{C}^n$. Then for $w \in \mathbb{C}^n$ and $0 < p < \infty$, we define

$$B_{\psi,p}(|u|)(w) = \int_{\mathbb{C}^n} |K_{\phi,w}(\psi(z))|^p |u(z)|^p e^{-p(\phi(z)+\phi(w))} dv(z).$$

Moreover, the results we obtain will be given in terms of certain measure, which we define next. Let ψ be an entire self-map of $\mathbb{C}^n$ and u be an entire map of $\mathbb{C}^n$. For each $p > 0$, we define a positive Borel measure $\mu_{\psi,p}$ on $\mathbb{C}^n$ by

$$\mu_{\psi,p}(E) = \int_{\psi^{-1}(E)} |u(z)|^p e^{-p\phi(z)} dv(z),$$

where E is a Borel subset of $\mathbb{C}^n$. Now we are ready to present our main results.

Theorem 3.1. *Let $0 < p \leq q < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. Then the operator $uC_\psi : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$ is bounded if and only if $\sup_{w\in\mathbb{C}^n} B_{\psi,q}(|u|)(w)$ is finite.*

Proof. First, suppose uC_ψ is bounded from $\mathcal{F}^p_\phi$ into $\mathcal{F}^q_\phi$. For a fixed $w \in \mathbb{C}^n$, set $f_w(z) = K_\phi(z,w)e^{-\phi(w)}$. Then using (3), we get $f_w \in \mathcal{F}^p_\phi$ and there exists a positive constant C_1 such that $\|f_w\|_{p,\phi} \leq C_1$. By boundedness of uC_ψ, there exists a positive constant C_2 such that

$$\|uC_\psi(f_w)\|^q_{q,\phi} \leq C_2\|f_w\|^q_{p,\phi} \leq C_1C_2.$$

Then,

$$\int_{\mathbb{C}^n} |f_w(\psi(z))|^q |u(z)|^q e^{-q\phi(z)} dv(z) \leq C_1C_2.$$

Taking the supremum over $w \in \mathbb{C}^n$, we get $\sup_{w\in\mathbb{C}^n} B_{\psi,q}(|u|)(w) < \infty$.

To prove the converse, for a fixed $w \in \mathbb{C}^n$, set $F_w(z) = K_\phi(z,w)e^{-\phi(z)}$. Using (2), for each $z \in B(w,r)$ and for some positive constant C we have

$$|F_w(z)|^q \geq Ce^{q\phi(w)}.$$

Now integrating both sides against the measure $\lambda_{\psi,q}$, where $d\lambda_{\psi,q}(z) = e^{q\phi(z)}d\mu_{\psi,q}(z)$, we get

$$\begin{aligned} C\lambda_{\psi,q}(B(w,r)) &\leq \int_{B(w,r)} |F_w(z)|^q e^{-q\phi(w)} d\lambda_{\psi,q}(z) \\ &\leq \int_{\mathbb{C}^n} |K_\phi(z,w)|^q e^{-q\phi(w)} d\mu_{\psi,q}(z) \\ &= \int_{\mathbb{C}^n} |K_{\phi,w}(\psi(z))|^q |u(z)|^q e^{-q(\phi(z)+\phi(w))} dv(z). \end{aligned}$$

Therefore, we get

$$C\lambda_{\psi,q}(B(w,r)) \leq B_{\psi,q}(|u|)(w). \tag{4}$$

By our hypothesis, $\sup\limits_{w\in\mathbb{C}^n} B_{\psi,q}(|u|)(w) < \infty$, and we get $\lambda_{\psi,q}(B(w,r))$ is bounded on $\mathbb{C}^n$. By Lemma 2.1, this is equivalent to $\lambda_{\psi,q}$ is a (p,q)-Fock Carleson measure. Therefore, by definition of Carleson measure, for any $f \in \mathcal{F}^p_\phi$ there exists a constant $C_3 > 0$ such that

$$\int_{\mathbb{C}^n} |f(z)|^q e^{-q\phi(z)} d\lambda_{\psi,q}(z) \leq C_3 \|f\|^q_{p,\phi}.$$

The last inequality equivalent to

$$\|uC_\psi(f)\|^q_{q,\phi} \leq C_3 \|f\|^q_{p,\phi},$$

which gives the boundedness of uC_ψ. □

As an immediate consequence of Theorem 3.1 and Lemma 2.1, we get the following corollary.

Corollary 3.2. Let $0 < p \leq q < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. Then the operator uC_ψ is bounded from $\mathcal{F}^p_\phi$ into $\mathcal{F}^q_\phi$ if and only if $\lambda_{\psi,q}$ is a (p,q)-Fock Carleson measure, where $d\lambda_{\psi,q}(z) = e^{q\phi(z)}d\mu_{\psi,q}(z)$.

The following theorem gives an estimate of $\|uC_\psi\|_{q,\phi}$ in terms of L^∞-norm of the integral transform $B_{\psi,q}(|u|)$.

Theorem 3.3. *Let $0 < p \leq q < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. If uC_ψ is bounded from $\mathcal{F}^p_\phi$ into $\mathcal{F}^q_\phi$, then there exist two positive constants C_1 and C_2 such that*

$$C_1 \|B_{\psi,q}(|u|)\|_{L^\infty} \leq \|uC_\psi\|^q_{q,\phi} \leq C_2 \|B_{\psi,q}(|u|)\|_{L^\infty}. \tag{5}$$

Proof. For a fixed $w \in \mathbb{C}^n$, set $f_w(z) = K_\phi(z,w)e^{-\phi(w)}$. It is clear that $f_w \in \mathcal{F}^p_\phi$ and there exists a positive constant C such that $\|f_w\|_{p,\phi} \leq C$.

Now, we have the following inequalities.

$$\begin{aligned} B_{\psi,q}(|u|)(w) &= \int_{\mathbb{C}^n} |f_w(\psi(z))|^q |u(z)|^q e^{-q\phi(z)} dv(z) \\ &= \|uC_\psi(f_w)\|_{q,\phi}^q \\ &\leq \|f_w\|_{p,\phi}^q \|uC_\psi\|_{q,\phi}^q \\ &\leq C^q \|uC_\psi\|_{q,\phi}^q. \end{aligned} \tag{6}$$

Taking the supremum over $w \in \mathbb{C}^n$, we get

$$\sup_{w\in\mathbb{C}^n} B_{\psi,q}(|u|)(w) \leq C^q \|uC_\psi\|_{q,\phi}^q,$$

which gives the left-hand side of condition (5).

Now, we prove the other side of condition (5). By Proposition 2.3 of [19], for each $r > 0$ there exists a positive constant C_2 such that if f is an entire function of $\mathbb{C}^n$, then we have

$$\sup_{z\in B(a,r)} |f(z)e^{-\phi(z)}|^p \leq C_2 \int_{B(a,2r)} |f(w)|^p e^{-p\phi(w)} dv(w). \tag{7}$$

Now consider the r-lattice $\{a_k\}$ in $\mathbb{C}^n$ and a positive integer N such that every point in $\mathbb{C}^n$ belongs to at most N sets in $\{B(a_k, 2r)\}$. Since $\bigcup_{k=1}^{\infty} B(a_k, r) = \mathbb{C}^n$, the covering property gives

$$\begin{aligned} \|uC_\psi(f)\|_{q,\phi}^q &= \int_{\mathbb{C}^n} |f(\psi(z))|^q |u(z)|^q e^{-q\phi(z)} dv(z) \\ &= \int_{\mathbb{C}^n} |f(z)|^q d\mu_{\psi,q}(z) = \int_{\mathbb{C}^n} |f(z)|^q e^{-q\phi(z)} d\lambda_{\psi,q}(z) \\ &\leq \sum_{k=1}^{\infty} \int_{B(a_k,r)} |f(z)e^{-\phi(z)}|^q d\lambda_{\psi,q}(z) \\ &\leq \sum_{k=1}^{\infty} \lambda_{\psi,q}(B(a_k,r)) \left(\sup_{z\in B(a_k,r)} |f(z)e^{-\phi(z)}|^p \right)^{q/p} \\ &\leq C_3 \sum_{k=1}^{\infty} \lambda_{\psi,q}(B(a_k,r)) \left(\int_{B(a_k,2r)} |f(w)e^{-\phi(w)}|^p dv(w) \right)^{q/p}, \end{aligned}$$

where the last inequality holds by using (7). Since $q/p \geq 1$, using the fact $\sum_{k=1}^{\infty} b_k^m \leq \left(\sum_{k=1}^{\infty} b_k\right)^m$ whenever $1 \leq m < \infty$ and $b_k \geq 0$ for all k, we obtain

$$\begin{aligned}\|uC_\psi(f)\|_{q,\phi}^q &\leq C_3 \sup_{k\geq 1} \lambda_{\psi,q}(B(a_k,r)) \left(\sum_{k=1}^{\infty} \int_{B(a_k,2r)} |f(w)e^{-\phi(w)}|^p dv(w)\right)^{q/p} \\ &\leq C_3 N^{q/p} \sup_{k\geq 1} \lambda_{\psi,q}(B(a_k,r)) \|f\|_{p,\phi}^q.\end{aligned}$$

Using inequality (4), we get the desired result. □

The following lemma is a basic fact about the compactness of weighted composition operators. The lemma was proved in a standard way in many references, see for example Proposition 3.11 [8] in the context of analytic function spaces and see Proposition 4.3 [19] in the context of entire function spaces. The proof of this lemma can be obtained by simple modifications of the proofs of Proposition 3.11 and Proposition 4.3 in [8] and [19], respectively. Hence, we omit the trivial details.

Lemma 3.4. *Let $0 < p, q \leq \infty$ and $dd^c\phi \simeq \omega_0$. Let ψ be an entire self-map of $\mathbb{C}^n$ and u is an entire function of $\mathbb{C}^n$ such that uC_ψ is a bounded from $\mathcal{F}_\phi^p$ into $\mathcal{F}_\phi^q$. Then uC_ψ is compact if and only if whenever $\{f_n\}$ is bounded sequence in $\mathcal{F}_\phi^p$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{C}^n$, then $\|uC_\psi(f_n)\|_{q,\phi} \to 0$.*

The next theorem characterizes the compactness of the weighted composition operator uC_ψ that maps $\mathcal{F}_\phi^p$ to $\mathcal{F}_\phi^q$ when $0 < p \leq q < \infty$.

Theorem 3.5. *Let $0 < p \leq q < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. Then the bounded operator $uC_\psi : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ is compact if and only if*

$$\lim_{|z|\to\infty} B_{\psi,q}(|u|)(z) = 0. \tag{8}$$

Proof. Suppose first that uC_ψ is compact. For any $z \in \mathbb{C}^n$, set $F_z(w) = K_\phi(z,w)e^{-\phi(z)}$. Then, by estimate (1), there exist two positive constants θ and C such that for all $z, w \in \mathbb{C}^n$ we have

$$|F_z(w)|e^{-\phi(w)} \leq Ce^{-\theta|z-w|}.$$

Therefore, F_z converges to 0 uniformly on any compact subset of $\mathbb{C}^n$ as $|z| \to \infty$. Hence, by compactness of uC_ψ and Lemma 3.4, we get that

$\lim_{|z|\to\infty} \|uC_\psi(F_z)\|_{q,\phi} = 0$. On the other hand, we know that

$$\begin{aligned}\|uC_\psi(F_z)\|_{q,\phi}^q &= \int_{\mathbb{C}^n} |F_z(\psi(w))|^q |u(w)|^q e^{-q\phi(w)} d(w)\\ &= B_{\psi,q}(|u|)(z),\end{aligned}$$

which gives the desired result.

Conversely, suppose that condition (8) holds. Using a similar argument as that in the proof of Theorem 3.1, there exists a positive constant C_1 such that

$$C_1 \lambda_{\psi,q}(B(z,r)) \le B_{\psi,q}(|u|)(z).$$

By our hypothesis we get that $\lambda_{\psi,q}(B(z,r)) \to 0$ as $|z| \to 0$. Therefore, by Lemma 2.2, we get that $\lambda_{\psi,q}$ is a vanishing (q,p)-Fock Carleson measure. Hence, by the definition of vanishing Carleson measure, for any bounded sequence $\{f_j\}$ in $\mathcal{F}_\phi^p$ that converges to 0 on compact subsets of $\mathbb{C}^n$ we have

$$\begin{aligned}0 &= \lim_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z) e^{-\phi(z)}|^q d\lambda_{\psi,q}(z)\\ &= \lim_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)|^q d\mu_{\psi,q}(z)\\ &= \lim_{j\to\infty} \int_{\mathbb{C}^n} |f_j(\psi(z))|^q |u(z)|^q e^{-q\phi(z)} dv(z)\\ &= \lim_{j\to\infty} \|uC_\psi(f_j)\|_{\psi,q}^q,\end{aligned}$$

which gives the compactness of the operator uC_ψ by consulting Lemma 3.4. This completes the proof. □

The following theorem characterizes boundedness and compactness of the operator uC_ψ that maps $\mathcal{F}_\phi^q$ to $\mathcal{F}_\phi^q$ when $0 < q < p < \infty$.

Theorem 3.6. *Let $0 < q < p < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. Then the following statements are equivalent.*

(1) *The operator $uC_\psi : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ is bounded;*
(2) *The operator $uC_\psi : \mathcal{F}_\phi^p \to \mathcal{F}_\phi^q$ is compact;*
(3) *$B_{\psi,q}(|u|) \in L^{p/(p-q)}(\mathbb{C}^n, dv)$.*

Proof. First, we prove that (1) is equivalent to (2). Suppose that uC_ψ is bounded. Then for any $f \in \mathcal{F}_\phi^p$, there exists a constant $C > 0$ such that

$\|uC_\psi(f)\|^q_{q,\phi} \le C\|f\|^q_{p,\phi}$. On the other hand,

$$\begin{aligned}\|uC_\psi(f)\|^q_{q,\phi} &= \int_{\mathbb{C}^n} |f(\psi(z))|^q |u(z)|^q e^{-q\phi(z)} dv(z) \\ &= \int_{\mathbb{C}^n} |f(z)|^q d\mu_{\psi,q}(z) \\ &= \int_{\mathbb{C}^n} |f(z)|^q e^{-q\phi(z)} d\lambda_{\psi,q}(z).\end{aligned}$$

This gives,

$$\int_{\mathbb{C}^n} |f(z)|^q e^{-q\phi(z)} d\lambda_{\psi,q}(z) \le C\|f\|^q_{p,\phi}. \tag{9}$$

The last inequality gives that $\lambda_{\psi,q}$ is a (p,q)-Fock Carleson measure. Therefore, by Lemma 2.3, this is equivalent to $\lambda_{\psi,q}$ is a vanishing (p,q)-Fock Carleson measure. Then for any bounded sequence $\{f_j\}$ in $\mathcal{F}^p_\alpha$ that converges to 0 uniformly on any compact subset of $\mathbb{C}^n$ as $j \to \infty$, we have

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} \left| f_j(z) e^{-\phi(z)} \right|^q d\lambda_{\psi,q}(z) = 0,$$

this gives $\lim\limits_{j\to\infty} \|uC_\psi(f_j)\|^q_{q,\phi} = 0$. Using Lemma 3.4, we get the compactness of uC_ψ. Since (2) always implies (1), we get the equivalence of (1) and (2).

Second, we prove that (1) is equivalent to (3). Suppose that uC_ψ is bounded. Then, inequality (9) gives that $\lambda_{\psi,q}$ is a (p,q)-Fock Carleson measure. Again using Lemma 2.3 we conclude that the Berezin transform $\tilde{\lambda}_{\psi,q}$ belongs to $L^{p/(p-q)}(\mathbb{C}^n, dv)$. On the other hand,

$$\begin{aligned}\tilde{\lambda}_{\psi,q}(z) &= \int_{\mathbb{C}^n} |K_{\phi,z}(w)|^q e^{-q(\phi(w)+\phi(z))} d\lambda_{\psi,q}(w) \\ &= \int_{\mathbb{C}^n} |K_{\phi,z}(w)|^q e^{-q\phi(w)} d\mu_{\psi,q}(w) \\ &= \int_{\mathbb{C}^n} |K_{\phi,z}(\psi(w))|^q |u(w)|^q e^{-q(\phi(w)+\phi(z))} dv(w) \\ &= B_{\psi,q}(|u|)(z),\end{aligned}$$

this gives the equivalence of (1) and (3). This completes the proof. □

The following theorem estimates the Fock-norm of the bounded operator $uC_\psi : \mathcal{F}^p_\phi \to \mathcal{F}^q_\phi$, when $0 < q < p < \infty$, in terms of $L^{p/(p-q)}$-norm of the integral transform $B_{\psi,q}(|u|)$.

Theorem 3.7. *Let $0 < q < p < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. If the operator uC_ψ from $\mathcal{F}^p_\phi$ into $\mathcal{F}^q_\phi$ is bounded then there exist two positive constants C_1 and C_2 such that*

$$C_2\|B_{\psi,q}(|u|)\|_{L^{p/(p-q)}} \le \|uC_\psi\|^q_{q,\phi} \le C_1\|B_{\psi,q}(|u|)\|_{L^{p/(p-q)}}.$$

Proof. First, using an argument similar to that in the proof of Theorem 3.3, for any $w \in \mathbb{C}^n$ and some constant $C > 0$ we have

$$B_{\psi,q}(|u|)(w) \le C^q \|uC_\psi\|_{q,\phi}^q.$$

Using Theorem 3.6, we get boundedness of uC_ψ is equivalent to $B_{\psi,q}(|u|) \in L^{p/(p-q)}(\mathbb{C}^n, dv)$. This implies

$$\|B_{\psi,q}(|u|)\|_{L^{p/(p-q)}} \le C_1 \|uC_\psi\|_{q,\phi}^q.$$

On the other hand, from estimate (2), for $z \in B(w,r)$ there exists a constant C_2 such that

$$|K_{\phi,w}(z)|^q e^{-q\phi(w)} \ge C_2 e^{q\phi(z)}.$$

Then,

$$\begin{aligned} C_2 \int_{B(w,r)} e^{q\phi(z)} d\mu_{\psi,q}(z) &\le \int_{B(w,r)} |K_{\phi,w}(z)|^q e^{-q\phi(w)} d\mu_{\psi,q}(z) \\ &\le \int_{\mathbb{C}^n} |K_{\phi,w}(z)|^q e^{-q\phi(w)} d\mu_{\psi,q}(z) \\ &= \int_{\mathbb{C}^n} |K_{\phi,w}(\psi(z))|^q |u(z)|^q e^{-q\phi(w)} e^{-q\phi(z)} dv(z) \\ &= B_{\psi,q}(|u|)(w). \end{aligned} \tag{10}$$

By Proposition 2.3 of [19], we have for any entire function f

$$|f(z)|^q e^{-q\phi(z)} \le C_3 \int_{B(z,r)} |f(w)|^q e^{-q\phi(w)} dv(w).$$

Integrating both sides against the measure $\mu_{\psi,q}$, then using Fubini's Theorem, the fact $\chi_{B(z,r)}(w) = \chi_{B(w,r)}(z)$ and inequality (10) we get

$$\begin{aligned} &\int_{\mathbb{C}^n} |f(z)|^q d\mu_{\psi,q}(z) \\ &\le C_3 \int_{\mathbb{C}^n} e^{q\phi(z)} \int_{\mathbb{C}^n} |f(w)|^q \chi_{B(z,r)}(w) e^{-q\phi(w)} dv(w) d\mu_{\psi,q}(z) \\ &\le C_3 \int_{\mathbb{C}^n} |f(w)|^q e^{-q\phi(w)} dv(w) \int_{\mathbb{C}^n} \chi_{B(w,r)}(z) e^{q\phi(z)} d\mu_{\psi,q}(z) \\ &\le C_3/C_2 \int_{\mathbb{C}^n} |f(w)|^q e^{-q\phi(w)} B_{\psi,q}(|u|)(w) dv(w). \end{aligned} \tag{11}$$

Now, applying Hölder's inequality with exponent $p/q > 1$, we get

$$\begin{aligned}&\int_{\mathbb{C}^n} |f(z)|^q d\mu_{\psi,q}(z)\\ &\leq C_3/C_2 \left(\int_{\mathbb{C}^n} |f(w)|^p e^{-p\phi(w)} dv(w) \right)^{q/p}\\ &\times \left(\int_{\mathbb{C}^n} B_{\psi,q}^{p/(p-q)}(|u|)(w) dv(w) \right)^{(p-q)/p}\\ &= C_3/C_2 \|f\|_{p,\phi}^q \|B_{\psi,q}(|u|)\|_{L^{p/(p-q)}}.\end{aligned}$$

This completes the proof. □

The following theorem characterizes the boundedness and compactness of $uC_\psi : \mathcal{F}_\phi^\infty \to \mathcal{F}_\phi^q$ in terms of the integral operator $B_{\psi,q}(|u|)$. The proof of this theorem is similar to that of Theorem 3.6, but this time we use Lemma 2.4 instead of Lemma 2.3. Therefore, we omit the trivial proof's details.

Theorem 3.8. *Let $0 < q < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. Then the following statements are equivalent.*

(1) *The operator $uC_\psi : \mathcal{F}_\phi^\infty \to \mathcal{F}_\phi^q$ is bounded;*
(2) *The operator $uC_\psi : \mathcal{F}_\phi^\infty \to \mathcal{F}_\phi^q$ is compact;*
(3) *$B_{\psi,q}(|u|) \in L^1(\mathbb{C}^n, dv)$.*

The following theorem estimates the Fock-norm of a bounded operator $uC_\psi : \mathcal{F}_\phi^\infty \to \mathcal{F}_\phi^q$, when $0 < q < \infty$, in terms of L^1-norm of the integral transform $B_{\psi,q}(|u|)$.

Theorem 3.9. *Let $0 < q < \infty$, let ψ be an entire self-map of $\mathbb{C}^n$, and let u be an entire function of $\mathbb{C}^n$. If the operator uC_ψ is bounded from $\mathcal{F}_\phi^\infty$ into $\mathcal{F}_\phi^q$ then there exist two positive constants C_1 and C_2 such that*

$$C_2 \|B_{\psi,q}(|u|)\|_{L^1} \leq \|uC_\psi\|_{q,\phi}^q \leq C_1 \|B_{\psi,q}(|u|)\|_{L^1}.$$

Proof. First, using an argument similar to that in the proof of Theorem 3.3, for any $w \in \mathbb{C}^n$ and some constant $C > 0$ we have

$$B_{\psi,q}(|u|)(w) \leq C^q \|uC_\psi\|_{q,\phi}^q.$$

Using Theorem 3.8, we have boundedness of uC_ψ is equivalent to $B_{\psi,q}(|u|) \in L^1(\mathbb{C}^n, dv)$. This implies

$$\|B_{\psi,q}(|u|)\|_{L^1} \leq C_1 \|uC_\psi\|_{q,\phi}^q.$$

On the other hand, using an argument similar to that in the proof of Theorem 3.7, for any $f \in \mathcal{F}_\phi^\infty$ there exists a constant C_2 such that

$$\begin{aligned}\int_{\mathbb{C}^n} |f(z)|^q d\mu_{\psi,q}(z) &\leq C_2 \int_{\mathbb{C}^n} |f(w)|^q e^{-q\phi(w)} B_{\psi,q}(|u|)(w) dv(w) \\ &\leq C_2 \|f\|_{\infty,\phi}^q \int_{\mathbb{C}^n} B_{\psi,q}(|u|)(w) dv(w) \\ &= C_2 \|f\|_{\infty,\phi}^q \|B_{\psi,q}(|u|)\|_{L^1}.\end{aligned}$$

This completes the proof. □

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References

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