A NOTE ON OI TORSION ABELIAN GROUPS

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Abstract. Let R be a commutative ring with unity and M_R be a nonzero unital right R-module. We say that M is an OI R-module if for each $x \in R$, $Mx = M$ implies x is invertible in R. We give a characterization of OI torsion abelian groups in terms of their direct summands.

1. Introduction

Throughout, R will denote a commutative ring with unity and M_R will denote a unitary right R-module. We will use M for M_R when the coefficient ring is obvious. Define $\rho_x \colon M_R \to M_R$, the right multiplication map by an element $x \in R$, by $\rho_x(m) = mx$ for all $m \in M$. Clearly, ρ_x is a module homomorphism. When ρ_x is surjective then $Mx = M$ and we say that M is *divisible by* x, written $x|M$. Also, A will denote an abelian group and we will write the group additively. We will use $\oplus$ to denote direct sums.

C. J. Maxson presented in [3] the following construction. If R is nonlocal then there exists noninvertible elements r and s such that $r + s = 1$. Suppose that $f : M_R \to M_R$ is a homogenous function (preserving scalar multiplication) and f is linear on submodules Mr and Ms. Calculations show that f will also be linear on M. A collection of proper submodules is said to *force linearity* if every homogeneous map which is linear on the collection of submodules is also linear on M. The *forcing linearity number* of M is the minimum integer n (if one exists) such that a collection of n proper submodules forces linearity on M. Thus, assuming that Mr and Ms are both proper submodules, then in this case, M will have forcing linearity number of at most two. Maxson asked if one can describe when right multiplication by a ring element *onto* a module implies that the element is invertible. Hence, in this case, if R satsified such a property then Mr and Ms would have to be proper submodules. To study this situation, the following terms are defined.

Definition 1.1. *Let $0 \neq M_R$ have the property that for all $x \in R$, if ρ_x is surjective, then x is invertible in R. Then M is called an OI R-module.*

Definition 1.2. *If every nonzero module of R is an OI module, then we say that R is an OI ring.*

If x is invertible in R, then the map ρ_x is an isomorphism for any R-module. One could consider OI modules as a generalization of hopfian modules, that is, the class of modules in which every epimorphism is an isomorphism. In fact, the term OI comes from "onto implies invertible." Also, if R is an OI ring and $x \in R$ such that there exists a nonzero module for which ρ_x is surjective, then ρ_x is an isomorphism, and hence it is surjective, on every R-module.

In [2], the class of OI rings is fully characterized. This paper begins the study of the connection between an OI module and its ring of scalars. We first consider modules over $\mathbb{Z}$, that is the class of abelian groups. Thus, we make the following definition.

Definition 1.3. *Let $0 \neq A$ be an abelian group with the property that for all $a \in A$, if ρ_a is surjective, then $a \in \{-1, 1\}$. Then A is an OI $\mathbb{Z}$-module, or, in other words, an OI group.*

2. Examples

Clearly, the class of divisible groups are not OI. Recall that all groups in this section are additively written abelian.

Example 2.1. *Let $0 \neq M$ be a $\mathbb{Z}_4$-module. Then, for any $m \in M$, we have $4m = 4(m\overline{1}) = m\overline{4} = 0$. So the order of m divides 4. Since M is a nonzero module, there must be at least one element of even order (not 1), so let y be an element of maximal even order. Thus, if $y = m \cdot \overline{2} \in M \cdot \overline{2}$, then m would have an even order greater than the order of y (since the order of y is half the order of m). Since this contradicts y having maximal even order, $y \notin M \cdot \overline{2}$. Therefore, $\rho_{\overline{2}}$ is not surjective. Clearly the zero map is not surjective. So only $\rho_{\overline{1}}$ and $\rho_{\overline{3}}$ could be surjective right multiplication maps. Since $\overline{1}$ and $\overline{3}$ are both invertible in $\mathbb{Z}_4$, M is an OI $\mathbb{Z}_4$-module and we have $\mathbb{Z}_4$ is an OI ring. A similar argument can be used to show that $\mathbb{Z}_{p^n}$ is an OI ring for any prime p.*

Example 2.2. *Let $\mathbb{Z}_n$ be a module over $\mathbb{Z}$. Then ρ_x is surjective if and only if $(x, n) = 1$ if and only if x is invertible in the ring $\mathbb{Z}_n$. Hence, $\mathbb{Z}_n$ is not an OI group because x does not have to be invertible in $\mathbb{Z}$. Note that $\mathbb{Z}_n$ is a reduced group, so in a sense, one does not think of OI groups as the opposite of divisible groups.*

3. OI Groups

In this section, it is more useful to use the convention of $n|A$ rather than ρ_n is surjective. The following two lemmas from [1] are provided with proof, the first of which allows us to concentrate on the case where n is prime.

Lemma 3.1. *For all $n \in \mathbb{Z}$, $n|A$ if and only if $p|A$ for every prime $p|n$.*

Proof. Suppose $n|A$. Let $x \in A$ and let $p|n$. If $x = ny$, then $x = p(qy)$ for some $q \in \mathbb{Z}$. Hence, $p|A$.

Now suppose that $n = p_1 \dots p_k$ where the primes are not necessarily distinct. Then $x = p_1y_1 = p_1(p_2y_2) = \cdots = p_1p_2 \dots p_{k-1}(p_ky_k) = ny_k$. Consequently, $n|A$. $\square$

Lemma 3.2. *Let $A = \oplus_i A_i$. For all $n \in \mathbb{Z}$, $n|A$ if and only if $n|A_i$ for every i.*

Proof. Suppose $n|A$ and let $a_i \in A_i$. Thus there exists $b \in A$ such that $a = nb$ and since A_i is a direct summand of A, $b \in A_i$. So, $n|A_i$.

Now suppose that $n|A_i$ for every i and let $a \in A$. Then $a = \sum a_i$ and for each i, $a_i = nb_i$ for some $b_i \in A_i$. Thus, $a = \sum nb_i = n \sum b_i$. Therefore, $n|A$. $\square$

Theorem 3.3. *Let A be a torsion abelian group. Then A is an OI group if and only if, for each prime p, A has a cyclic direct summand $\mathbb{Z}_{p^n}$ for some n.*

Proof. Let A be an OI group and let A_p be the p-primary component of A, that is, $A_p = \{x \in A | p^n x = 0 \text{ for some } n \geq 0\}$. It is well-known that $A = \oplus_p A_p$ [1]. Decompose $A_p = B \oplus D$ where B is reduced and D is the divisible subgroup of A_p. Since A is OI, A_p is OI and so A_p is not divisible by any prime. Thus, A_p cannot be divisible and so $B \neq \{0\}$. By Lemma 10.34 in [4], B contains a pure nonzero cyclic subgroup C. Since A is torsion, we can take $C = \mathbb{Z}_{p^n}$ for some n. By Corollary 3.41 from [4], a pure subgroup of bounded order is a direct summand. Consequently, C is a direct summand of B and so also of A_p and finally of A.

Conversely, suppose A has a $\mathbb{Z}_{p^n}$ direct summand for some n for each prime p. If $p|A$ then p divides every direct summand of A. Since $p \nmid \mathbb{Z}_{p^n}$, then by Lemma 1, $n \nmid A$ for every n divisible by p. So, if $n|A$, then $n \in \{-1, 1\}$ and A is an OI group. $\square$

Further research can either continue the investigation into more classes of abelian groups or extend these structure theorems to torsion modules over rings in general.

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