SMITH NUMBERS FROM PRIMES WITH SMALL DIGITS

PATRICK COSTELLO

Abstract. We find three new ways to construct Smith numbers from primes that contain small digits and a known digit sum. The resulting Smith numbers may have a fairly large number of digits.

1. Introduction

In 1982, Albert Wilansky, a mathematics professor at Lehigh University wrote a short article in the *Two-Year College Mathematics Journal* [9]. In that article he identified a new type of natural number. He defined a *Smith number* to be a composite number where the sum of the digits in its prime factorization is equal to the digit sum of the number. The set was named in honor of Wilansky's brother-in-law, Dr. Harold Smith, whose telephone number 493-7775 when written as the single number $4,937,775$ possessed this interesting characteristic. Adding the digits in the number and the digits of its prime factors 3, 5, 5, and $65,837$ resulted in identical sums of 42.

Since that time, many things have been discovered about Smith numbers including two different infinite sequences of Smith numbers [5, 7]. Both of these sequences depend on knowing how to factor the number $10^n - 1$ in order to construct the actual Smith number. Since factoring is a difficult operation, these sequences provide little help in the attempt to create actual large Smith numbers. A Smith number with over two million digits was produced by Samuel Yates in [12]. Using a repunit and large palindromic prime, Yates was able to produce Smith numbers having ten million digits and thirteen million digits [13]. Using the same repunit and a larger palindromic prime found by Heuer [2], the current author was able to find a Smith number with 32 million digits [3]. Shortly after that article, a larger palindromic prime was found by Heuer and a 107 million digit Smith was announced on the author's web page [4].

While many efforts were made to create very large Smith numbers, there were some efforts directed toward finding newer ways to construct Smith numbers. Amin Witno [11] gives two ways to construct Smith numbers from primes with a digit sum of 5. He challenges the reader to find other constructions. "We leave as a challenge to the reader to find other multipliers m for which the number mP is Smith for every prime P of digital sum $S(P) = 5$ or of a fixed other digital sum, preferably small." Three such results are presented in this paper. These results can then be used to generate large Smith numbers.

2. Notation and Basic Facts

For any positive integer n, we let $S(n)$ denote the sum of the digits of n. For any positive integer n, we let $S_p(n)$ denote the sum of the digits of the prime factorization of n. For example, $S(27) = 2 + 7 = 9$ and $S_p(27) = S_p(3 \cdot 3 \cdot 3) = 3 + 3 + 3 = 9$. Hence, 27 is a Smith number. Two of the well-known properties of the two functions are $S(10^k n) = S(n)$ and $S_p(mn) = S_p(m) + S_p(n)$ for natural numbers m and n.

A *repunit*, denoted R_n, is a number consisting of a string of n one's. For example, $R_4 = 1111$. Currently, the largest known prime repunit is R_{1031} which was shown to be prime by Hugh Williams and Harvey Dubner in 1986 [10]. The repunits R_{49081}, R_{86453}, R_{109297}, and R_{270343} are all probable primes, but have not been proven prime [1]. Repunits are often used in the construction of large Smith numbers because of the following facts.

<u>Fact 1</u>. If you multiply $9R_n$ by any natural number less than $9R_n$, then the digit sum is $9n$, i.e.,

$$S(M \cdot 9R_n) = 9n = S(9R_n), \text{ when } M < 9R_n.$$

For a proof see [6].

For example,

$$S(13 \cdot 99) = S(1287) = 1 + 2 + 8 + 7 = 18 = S(99).$$

Keith Wayland and Sham Oltikar in [8] provided the following fact.

<u>Fact 2</u>. If $S(u) > S_p(u)$ and $S(u) \equiv S_p(u) \pmod 7$, then $10^k \cdot u$ is a Smith number where $k = (S(u) - S_p(u))/7$.

For example, when $u = 49995 = 5 \cdot 9999 = 3^2 \cdot 5 \cdot 11 \cdot 101$, you have $S(u) = 36$ (by Fact 1) and $S_p(u) = 15$ which means that $S(u) \equiv S_p(u)$ (mod 7) and $k = 3$. Hence, 49995000 is a Smith number.

3. New Rules for Finding Smith Numbers

Amin Witno [11] found two new forms of Smith numbers involving primes with a digit sum of 5. He issued a challenge to find other forms. We prove three more forms that Smith numbers can take involving primes with a certain digit sum.

The two forms for Smith numbers that Witno established are:

1) $21P$ where $P > 5$ is a prime and $S(P) = 5$,
2) $112P$ where $P > 5$ is a prime and $S(P) = 5$.

He was able to find many primes P with digit sum 5 and generate the corresponding multiples that give a Smith number. For example, when $P = 23$, form 1) gives that $21 \cdot 23 = 483$ is a Smith number and form 2) gives that $112 \cdot 23 = 2576$ is a Smith number.

For the first new form, we generate Smith numbers from primes with small digits and digit sum 13.

Theorem 1. *Let P be a prime containing only digits 0, 1, 2, 3 and digit sum 13. Then $11001P$ is a Smith number.*

Proof. Let P be a prime containing only digits 0, 1, 2, 3 and digit sum 13. Then $S(11001P) = S(10000P + 1000P + P)$. Since the largest digit in P is at most 3, the sum $10000P + 1000P + P$ will never cause a carry and so $S(1000P + 1000P + P) = S(1000P) + S(1000P) + S(P)$. Using that $S(10^n P) = S(P)$, we get that $S(11001P) = 3S(P) = 3 * 13 = 39$. On the other hand, $S_p(11001P) = S_p(3 \cdot 19 \cdot 193P) = 3 + 10 + 13 + 13 = 39$. Hence, $S(11001P) = S_p(11001P)$ and $11001P$ is a Smith number. □

For example, let $P = 33331$. This is a prime satisfying the conditions of Theorem 1. The product $11001P = 366674331$ is then a Smith number. While there are only five 5-digit primes satisfying the conditions of Theorem 1, there are fifty 6-digit primes satisfying the conditions. A somewhat larger example is produced from the prime $P = 3333 \cdot 10^{43} + 1$ (said to be prime by the ProvablePrimeQ function in *Mathematica*). The product $11001P$ is then a 51-digit Smith number.

For the second new form, we generate Smith numbers from primes with small digits and digit sum 11.

Theorem 2. *Let P be a prime containing only digits 0, 1, 2, 3 and digit sum 11. Then $101001P$ is a Smith number.*

Proof. Let P be a prime containing only digits 0, 1, 2, 3 and digit sum 11. Then $S(101001P) = S(100000P + 1000P + P)$. Since the largest digit in P is at most 3, the sum $100000P + 1000P + P$ will never cause a carry and so $S(100000P + 1000P + P) = S(100000P) + S(1000P) + S(P)$. Using that $S(10^n P) = S(P)$, we get that $S(101001P) = 3S(P) = 3 \cdot 11 = 33$. On the other hand, $S_p(101001P) = S_p(3 \cdot 131 \cdot 257P) = 3 + 5 + 14 + 11 = 33$. Hence, $S(101001P) = S_p(101001P)$ and $101001P$ is a Smith number. □

For example, let $P = 2333$. This is a prime satisfying the conditions of Theorem 2. The product $101001P = 235635333$ is then a Smith number. While there are only two 4-digit primes satisfying the conditions of Theorem 2, there are sixteen 5-digit primes and ninety-eight 6-digit primes satisfying the conditions. A somewhat larger example is produced from the prime $P = 2233 \cdot 10^{64} + 1$ (said to be prime by the ProvablePrimeQ function in *Mathematica*). The product $101001P$ is a 73-digit Smith number.

For the third new form, we generate Smith numbers from primes with digits less than 5 and digit sum congruent to 2 (mod 7).

Theorem 3. *Let P be a prime greater than 101 containing only digits 0, 1, 2, 3, 4 and digit sum $7n+2$ for some positive integer n. Then $10^n \cdot 11P$ and $10^n \cdot 101P$ are Smith numbers.*

Proof. Let P be a prime greater than 101 containing only digits 0, 1, 2, 3, 4 and digit sum $7n + 2$. Then $S(11P) = S(10P + P)$ and $S(101P) = S(100P + P)$. Since the largest digit in P is at most 4, the sums $10P + P$ and $100P + P$ will never cause a carry and so $S(11P) = S(10P) + S(P) = 2S(P) = 2(7n + 2) = 14n + 4$. In the same manner, $S(101P) = 14n + 4$. On the other hand, $S_p(11P) = Sp(101P) = 2 + (7n + 2) = 7n + 4$. Then $(S(11P) - Sp(11P))/7 = (7n)/7 = n$ and so $10^n \cdot 11P$ is a Smith number by Fact 2. Similarly, $10^n \cdot 101P$ is a Smith number by Fact 2. □

Note that when $n \equiv 1 \pmod 3$, the expression $7n + 2 \equiv 0 \pmod 3$. Hence, if P has digit sum $7n + 2$ it will be divisible by 3 and not prime. As an example of Theorem 3, let $P = 32443$. This is a prime containing digits less than 5 and digit sum 16 (where $n = 2$). Then the number $11 \cdot 10^2 P = 35687300$ is a Smith number. The advantage of this theorem over the previous two theorems is that since the size of the digit sum can grow, one is not forced to introduce lots of zeros to get a large prime P. In fact, the current largest repunit $R_{1031} = 111\ldots1$ with 1031 ones has a digit sum of $1031 = 7 * 147 + 2$ and so Theorem 3 says $10^{147} \cdot 11R_{1031}$ and $10^{147} \cdot 101R_{1031}$ are Smith numbers. The other four probably prime repunits do not have digit sums congruent to 2 (mod 7).

We can construct a few fairly large Smith numbers using primes that have lots of repeating nonzero digits. We do this by looking for primes having blocks of 7 copies of the repeating digit and then adjusting the final digits so the digit sum is congruent to 2 (mod 7). We found an 84-digit prime with 83 initial ones. According to the ProvablePrimeQ function in *Mathematica*, the value $P = R_{84} + 2 = 1111\ldots113$ is a prime. Its digit sum is $7 \cdot 12 + 2$ and so Theorem 3 says that $10^{12} \cdot 11P$ and $10^{12} \cdot 101P$ are Smith numbers. We found a 273-digit prime with 268 initial twos. According to the ProvablePrimeQ function in *Mathematica*, the value $P = 2R_{273} + 10001 = 2222\ldots232223$ is a prime. Its digit sum is $2 \cdot 273 + 2 = 7 \cdot 78 + 2$ and so Theorem 3 says that $10^{78} \cdot 11P$ and $10^{78} \cdot 101P$ are Smith numbers. We found a 110-digit prime with 105 initial threes. According to the ProvablePrimeQ function in *Mathematica*, the value $P = 3R_{105} \cdot 10^5 + 44413 = 3333\ldots3344413$ is a prime. Its digit sum is $3 \cdot 105 + 16 =$

$7 \cdot 47 + 2$ and so Theorem 3 says that $10^{47} \cdot 11P$ and $10^{47} \cdot 101P$ are Smith numbers. We found a 238-digit prime with 236 initial fours. According to the ProvablePrimeQ function in *Mathematica*, the value $P = 4R_{238} - 23 = 4444 \ldots 4421$ is a prime. Its digit sum is $4 \cdot 238 - 5 = 7 \cdot 135 + 2$ and so Theorem 3 says that $10^{135} \cdot 11P$ and $10^{135} \cdot 101P$ are Smith numbers.

For somewhat smaller Smith numbers using Theorem 3, there are ninety-two 6-digit primes with digits less than 5 and digit sum 16. There are 677 7-digit primes with digits less than 5 and digit sum 16 ($n = 2$) and 44 7-digit primes with digits less than 5 and digit sum 23 ($n = 3$).

4. Conclusions

Following the ideas of Witno, three new rules for constructing Smith numbers from primes with a certain digit sum were developed. The last rule allows for constructing fairly large Smith numbers.

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Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, KY 40475
E-mail address: Pat.Costello@eku.edu