WHEN A MATRIX AND ITS INVERSE ARE NONNEGATIVE

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ABSTRACT. In this article we prove that A and A^{-1} are stochastic if and only of A is a permutation matrix. Then we extend this result to show that A and A^{-1} are nonnegative if and only if it is a product of a diagonal matrix with all positive diagonal entries and a permutation matrix.

1. INTRODUCTION

Applications abound with nonnegative matrices. For example, the discrete Laplacian leads to a nonnegative matrix. The matrix $\exp(At)$ that defines the solution of the system of differential equations is nonnegative in some applications. The system of difference equations $p(k) = A\,p(k-1)$ has a nonnegative coefficient matrix A in many applications. Nonnegative matrices are so pervasive that any result of nonnegative matrices should be interesting.

A short proof of the fact that A and A^{-1} are stochastic matrices if and only if A is a permutation matrix is given in [1]. Here we present another proof of this fact. This proof is longer, but shows the power of canonical forms of stochastic matrices. Then we extend this result to show that A and A^{-1} are nonnegative if and only if it is a product of a diagonal matrix with all positive diagonal entries and a permutation matrix.

A matrix is called stochastic if it is a nonnegative matrix for which each of its row sums equals 1. Clearly, if A is a permutation matrix, then A and A^{-1} are stochastic. Now we prove in three steps that if A and A^{-1} are stochastic, then A is a permutation matrix. In Section 2 we state a key spectral property of A when A and A^{-1} are stochastic. In Section 3 we mention a canonical form of a stochastic matrix. In Section 4 we develop a canonical form of A when A and A^{-1} are stochastic. This canonical form immediately gives us the result that if A and A^{-1} are stochastic, then A is a permutation matrix. Finally, in Section 5 we extend this result to the case when A and A^{-1} are nonnegative.

2. Spectral Property

Here we state a key spectral property of A when A and A^{-1} are stochastic.

Theorem 2.1. *If A and A^{-1} are stochastic, then all eigenvalues of A lie on the unit circle in the complex plane.*

Proof. We use the fact [3] that all eigenvalues of a stochastic matrix A are on the closed unit disk $\{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$. Suppose A has eigenvalues which are inside the unit circle $\{\lambda \in \mathbf{C} : |\lambda| = 1\}$. Then A^{-1} has eigenvalues which are outside the unit circle. But that is impossible because A^{-1} is stochastic. So all eigenvalues of A must be on the unit circle. □

3. Canonical Form of Stochastic Matrices

What we discuss in this section can be found in [2] or [3]. But for the convenience of readers, we present it here. If a stochastic matrix A is reducible, then, by definition, there exists a permutation matrix P and square matrices X and Z such that

$$P^T AP = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}.$$

We denote this by writing

$$A \sim \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}.$$

If X or Z is reducible, then another symmetric permutation can be performed to produce

$$\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \sim \begin{bmatrix} R & S & T \\ 0 & U & V \\ 0 & 0 & W \end{bmatrix},$$

where R, U, and W are square. Repeating this process eventually yields

$$A \sim \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ 0 & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{mm} \end{bmatrix},$$

where each X_{ii} is irreducible. Finally, if there exist rows having nonzero entries only in diagonal blocks, then symmetrically permute all such rows to the bottom to produce

$$A \sim \left[\begin{array}{cccc|cccc} A_{11} & A_{12} & \cdots & A_{1r} & A_{1,r+1} & A_{1,r+2} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2r} & A_{2,r+1} & A_{2,r+2} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} & A_{r,r+1} & A_{r,r+2} & \cdots & A_{rm} \\ \hline 0 & 0 & \cdots & 0 & A_{r+1,r+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & A_{r+2,r+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A_{mm} \end{array}\right], \quad (1)$$

where each A_{ii} is irreducible for $1 \le i \le m$. The form given on the right-hand side of (1) is called the canonical form of a stochastic matrix A.

Now we mention an important spectral property of A_{ii} for $1 \le i \le r$ in the canonical form (1).

Theorem 3.1. *In the canonical form (1), we have $\rho(A_{ii}) < 1$ for $1 \le i \le r$. Here $\rho(A_{ii})$ denotes the spectral radius of A_{ii}.*

Proof. This is clearly true when A_{ii} is 1×1. So suppose that the order of A_{ii} is at least 2. Because there must be at least one A_{ij}, with $i < j$, which is nonnegative and not zero, it follows that

$$A_{ii}e \le e \quad \text{and} \quad A_{ii}e \ne e,$$

where e is the vector of all 1's. It is clear that $\rho(A_{ii}) \le 1$. Suppose $\rho(A_{ii}) = 1$. Let $y > 0$ be the left Perron vector of A_{ii} so that $y^T A_{ii} = y^T$ and let $x = e - A_{ii}e \ge 0$. Since $A_{ii}e \ne e$, x has a positive component. So $y^T x > 0$. On the other hand,

$$y^T x = y^T(e - A_{ii}e) = y^T e - y^T A_{ii} e = y^T e - y^T e = 0,$$

which is a contradiction. So we see that $\rho(A_{ii}) < 1$. $\square$

4. Canonical Form When A and A^{-1} are Stochastic

Throughout this section we assume that A and A^{-1} are stochastic. Since all eigenvalues of A are on the unit circle by Theorem 2.1 and $\sigma(A) = \bigcup_{i=1}^m \sigma(A_{ii})$, Theorem 3.1 implies that $r = 0$ in (1), and hence the canonical form (1) reduces to

$$A \sim \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix}, \quad (2)$$

where each A_{kk} for $1 \le k \le m$ is irreducible.

Now we find further structure in (2). We divide A_{kk} for $k = 1, 2, \ldots, m$ into two groups. Let the first group be made of all A_{kk} which are 1×1, and the second group all A_{kk} whose orders are greater than 1. Without loss of generality, we may assume that A_{kk} are 1×1 for $1 \leq k \leq s$ and that the orders of A_{kk} are greater than 1 for $s+1 \leq k \leq t$ with $s + t = m$.

Since each A_{kk} is stochastic, we have

$$A_{kk} = 1 \quad \text{for} \quad 1 \leq k \leq s. \tag{3}$$

Now consider A_{kk} with $s+1 \leq k \leq m$. Observe that all the eigenvalues of A_{kk} are on the unit circle. Since the order of A_{kk} is greater than 1 and A_{kk} is irreducible, it has more than one eigenvalue on the unit circle. So A_{kk} is an imprimitive matrix.

Let us recall the following Frobenius canonical form for imprimitive matrices [4].

Theorem 4.1. *For each imprimitive matrix B with index of imprimitivity $h \geq 2$,*

$$B \sim \begin{bmatrix} 0 & B_{12} & 0 & \cdots & 0 \\ 0 & 0 & B_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & B_{h-1,h} \\ B_{h1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where the zero blocks on the main diagonal are square. Recall that $\sim$ denotes permutation similarity.

Suppose that the order of the imprimitive stochastic matrix A_{kk} is h. Since all eigenvalues of A_{kk} are on the unit circle and they are all simple, it follows that the index of imprimitivity of A_{kk} is also h. Hence, using Theorem 4.1 with $B = A_{kk}$ and using the fact that A_{kk} is stochastic we see that, for $s+1 \leq k \leq m$,

$$A_{kk} \sim \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \equiv P_k. \tag{4}$$

Using (3) and (4), we see that (2) further reduces to

$$A \sim \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & P_{s+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & P_{s+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & P_t \end{array}\right], \tag{5}$$

where the (1,1)-block is the $s \times s$ identity matrix. Since each P_k for $s+1 \leq k \leq t$ is a permutation matrix, it follows that A is a permutation matrix. In summary we have the following result.

Theorem 4.2. *A matrix and its inverse are stochastic if and only if it is a permutation matrix.*

5. Extension to Nonnegative Matrices

We extend our result from stochastic matrices to nonnegative matrices. In fact we have the following theorem.

Theorem 5.1. *A matrix and its inverse are nonnegative matrices if and only if it is the product of a diagonal matrix with all positive diagonal entries and a permutation matrix.*

Proof. Suppose that $A = DP$, where D is a diagonal matrix with all positive diagonal entries and P is a permutation matrix. Then clearly both A and $A^{-1} = P^{-1}D^{-1}$ are nonnegative matrices.

Conversely, suppose that A and A^{-1} are nonnegative matrices. Since A is invertible and nonnegative, each row of A has at least one positive entry. So if we let D be a diagonal matrix with each diagonal entry the sum of the corresponding row of A, then D is a diagonal matrix with all positive diagonal entries. Observe that if we let $P = D^{-1}A$, then P is an invertible stochastic matrix. Since $P^{-1} = A^{-1}D$ is nonnegative and

$$e = Ie = P^{-1}Pe = P^{-1}e,$$

P^{-1} is a stochastic matrix. So by Theorem 4.2 P is a permutation matrix, and $A = DP$. □

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MSC2010: 15A18

Key words and phrases: stochastic matrix, permutation matrix, nonnegative matrix

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