INVARIANTS OF STATIONARY AF-ALGEBRAS AND TORSION SUBGROUPS OF ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION

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Abstract. Let G_A be an AF-algebra given by a periodic Bratteli diagram with the incidence matrix $A \in GL(n, \mathbb{Z})$. For a given polynomial $p(x) \in \mathbb{Z}[x]$ we assign to G_A a finite abelian group $Ab_{p(x)}(G_A) = \mathbb{Z}^n/p(A)\mathbb{Z}^n$. It is shown that if $p(0) = \pm 1$ and $\mathbb{Z}[x]/\langle p(x)\rangle$ is a principal ideal domain, then $Ab_{p(x)}(G_A)$ is an invariant of the strong stable isomorphism class of G_A. For $n = 2$ and $p(x) = x - 1$ we conjecture a formula linking values of the invariant and torsion subgroup of elliptic curves with complex multiplication.

1. Introduction

Let $A \in GL(n, \mathbb{Z})$ be a strictly positive integer matrix and consider the following two objects, naturally attached to A. The first one, which we denote by (G_A, σ_A), is a pair consisting of an AF-algebra, G_A, given by an infinite periodic Bratteli diagram with the incidence matrix A and a shift automorphism, σ_A, canonically attached to G_A. (The definitions of an AF-algebra, a Bratteli diagram, and a shift automorphism are given in Section 2.) The second object is an abelian group, which can be introduced as follows. Let $p(x) \in \mathbb{Z}[x]$ be a polynomial over $\mathbb{Z}$, such that $p(0) = \pm 1$ and $\mathbb{Z}[x]/\langle p(x)\rangle$ is a principal ideal domain; here $\langle p(x)\rangle$ means the ideal generated by $p(x)$. Notice that $\mathbb{Z}[x]/\langle p(x)\rangle$ is a principal ideal domain whenever $p(x)$ is an irreducible polynomial and roots of $p(x)$ generate an algebraic number field whose ring of integers is a principal ideal domain. Consider the following abelian group:

$$\mathbb{Z}^n/p(A)\mathbb{Z}^n := Ab_{p(x)}(G_A), \tag{1}$$

which we shall call an *abelianized* G_A at the polynomial $p(x)$. Recall that the AF-algebras G_A and $G_{A'}$ are said to be stably isomorphic, whenever $G_A \otimes \mathcal{K} \cong G_{A'} \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$.

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Definition 1. *The AF-algebras G_A and $G_{A'}$ are said to be strongly stably isomorphic if they are stably isomorphic and $\sigma_A, \sigma_{A'}$ are the conjugate shift automorphisms.*

Roughly speaking, the stable isomorphism is a property of AF-algebra G_A, while the strong stable isomorphism is a property of the AF-algebra G_A along with its incidence matrix A. The main result of the present note is the following theorem.

Theorem 1. *For each polynomial $p(x) \in \mathbb{Z}[x]$, such that $p(0) = \pm 1$ and $\mathbb{Z}[x]/\langle p(x)\rangle$ is a principal ideal domain, the abelian group $Ab_{p(x)}(G_A)$ is an invariant of the strong stable isomorphism class of the AF-algebra G_A.*

Remark 1. The reader can find many more numerical invariants of stationary AF-algebras in the remarkable monograph by Bratteli, Jorgensen & Ostrovsky [2]; notice that the authors consider the case when A is not necessarily a unimodular matrix.

Let E_{CM} be an elliptic curve with complex multiplication by an order of conductor $f \geq 1$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, where $d \neq 1$ [12, p. 96]. Consider a periodic continued fraction $f\omega = [a_0, \overline{a_1, \dots, a_n}]$, where $\omega = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \pmod 4$ and $\omega = \sqrt{d}$ if $d \equiv 2, 3 \pmod 4$. We shall introduce an integer matrix $A = \prod_{i=1}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$, see Section 4.1 for a motivation.

Conjecture 1. ("Weil's Conjecture for torsion points") *For each E_{CM} there exists a number field K such that $E_{CM} \cong E(K)$ and a twist of $E(K)$ such that $E_{tors}(K) \cong Ab_{x-1}(G_A)$, where $E_{tors}(K)$ is the torsion subgroup of $E(K)$.*

Remark 2. Conjecture 1 is an analog of (one of) classical Weil's Conjectures for projective varieties over finite fields [4, pp. 449–451]; indeed, it identifies $E_{tors}(K)$ with the fixed points of an automorphism A of the cohomology group $H^1(E(K); \mathbb{Z})$, see also the last paragraph of Section 3.

The note is organized as follows. The preliminary facts are brought together in Section 2. Theorem 1 is proved in Section 3. In Section 4 conjecture 1 is explained and some examples are given.

2. Preliminaries

An *AF-algebra* (approximately finite-dimensional C^*-algebra) is defined to be the norm closure of an ascending sequence of the finite-dimensional C^*-algebras M_n's, where M_n is the C^*-algebra of the $n \times n$ matrices with the entries in $\mathbb{C}$. Here the index $n = (n_1, \dots, n_k)$ represents a semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots$, where M_i are the finite dimensional C^*-algebras and φ_i the homomorphisms between such algebras. The set-theoretic limit $\mathcal{A} = \lim M_i$ has a natural algebraic structure given by the formula $a_m + b_k \to a + b$; here $a_m \to a, b_k \to b$ for the sequences $a_m \in M_m, b_k \in M_k$. The homomorphisms φ_i can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \cdots \oplus M_{i_k}$ and $M_{i'} = M_{i'_1} \oplus \cdots \oplus M_{i'_k}$ be the semi-simple C^*-algebras and $\varphi_i \colon M_i \to M_{i'}$ the homomorphism. One has the two sets of vertices $V_{i_1}, \dots, V_{i_k}$ and $V_{i'_1}, \dots, V_{i'_k}$ joined by the a_{rs} edges, whenever the summand M_{i_r} contains a_{rs} copies of the summand $M_{i'_s}$ under the embedding φ_i. As i varies, one obtains an infinite graph called a *Bratteli diagram* of the AF-algebra [1]. The Bratteli diagram defines a unique AF-algebra.

If the homomorphisms $\varphi_1 = \varphi_2 = \cdots = Const$ in the definition of the AF-algebra $\mathcal{A}$, the Bratteli diagram of AF-algebra $\mathcal{A}$ is called *stationary*; by an abuse of notation, we shall refer to the corresponding AF-algebra as stationary as well. The stationary Bratteli diagram looks like a periodic graph with the incidence matrix $A = (a_{rs})$ repeated over and over again. Since matrix A is a non-negative integer matrix, one can take a power of A to obtain a strictly positive integer matrix – which we always assume to be the case. We shall denote the above AF-algebra by G_A. Recall that in the case of AF-algebras, the abelian monoid $V_{\mathbb{C}}(\mathcal{A})$ of finitely-generated projective modules over $\mathcal{A}$ (and a scale) defines the AF-algebra up to an isomorphism and is known as a *dimension group* of $\mathcal{A}$. We shall use a standard dictionary existing between the AF-algebras and their dimension groups [10, Section 7.3]. Instead of dealing with the AF-algebra G_A, we shall work with its dimension group $(K_0(G_A), K_0^+(G_A))$, where $K_0(G_A)$ is the lattice and $K_0^+(G_A)$ is a positive cone inside the lattice given by a sequence of the simplicial dimension groups:

$$\mathbb{Z}^n \xrightarrow{\text{A}} \mathbb{Z}^n \xrightarrow{\text{A}} \mathbb{Z}^n \xrightarrow{\text{A}} \cdots . \tag{2}$$

(The above notation comes from the K_0-group of G_A [10, p. 122].) There exists a natural automorphism, σ_A, of the dimension group $(K_0(G_A), K_0^+(G_A))$ [3, p. 37]. It can be defined as follows. Let $\lambda_A > 1$ be the Perron-Frobenius eigenvalue and $v_A = (v_A^{(1)}, \dots, v_A^{(n)}) \in \mathbb{R}^n_+$ the corresponding eigenvector of the matrix A. It is known that $K_0^+(G_A)$ is defined by the inequality $\mathbb{Z}v_A^{(1)} + \cdots + \mathbb{Z}v_A^{(n)} \geq 0$ and one can multiply $\mathbb{Z}$-module $\mathbb{Z}v_A^{(1)} + \cdots + \mathbb{Z}v_A^{(n)}$ by λ_A. It is easy to see that such a multiplication defines an automorphism of the dimension group $(K_0(G_A), K_0^+(G_A))$. The automorphism is called a *shift automorphism* and denoted by σ_A. The shift automorphisms $\sigma_A, \sigma_{A'}$

are said to be conjugate, if $\sigma_A \circ \theta = \theta \circ \sigma_{A'}$ for some order-isomorphism θ between the dimension groups $(K_0(G_A), K_0^+(G_A))$ and $(K_0(G_{A'}), K_0^+(G_{A'}))$. We shall write this fact as $(G_A, \sigma_A) \cong (G_{A'}, \sigma_{A'})$ (an isomorphism).

Lemma 1. *The pairs (G_A, σ_A) and $(G_{A'}, \sigma_{A'})$ are isomorphic if and only if the matrices A and A' are similar.*

Proof. By Theorem 6.4 of [3], $(G_A, \sigma_A) \cong (G_{A'}, \sigma_{A'})$ if and only if the matrices A and A' are shift equivalent, see [14] for a definition of the shift equivalence. On the other hand, since the matrices A and A' are unimodular, the shift equivalence between A and A' coincides with a similarity of the matrices in the group $GL(n, \mathbb{Z})$ [14, Corollary 2.13]. $\square$

Corollary 1. *The AF-algebras G_A and $G_{A'}$ are strongly stably isomorphic if and only if the matrices A and A' are similar.*

Proof. By a dictionary between the dimension groups and AF-algebras, the order-isomorphic dimension groups correspond to the stably isomorphic AF-algebra [3, Theorem 2.3]. Since σ_A and $\sigma_{A'}$ are conjugate, one gets a strong stable isomorphism. $\square$

Example 1. Let us show that Theorem 1 is non-trivial and the condition strong stable isomorphism cannot be relaxed to just stable isomorphism. Consider the unimodular matrices

$$A = \begin{pmatrix} a & a-1 \\ 1 & 1 \end{pmatrix} \text{ and } A_h = \begin{pmatrix} a-h & (a-h)(h+1)-1 \\ 1 & h+1 \end{pmatrix}, \tag{3}$$

where $a, h \in \mathbb{Z}$ and $a > h \geq 1$. Because eigenvalues of A and A_h coincide, one concludes that $(K_0(G_A), K_0^+(G_A)) \cong (K_0(G_{A_h}), K_0^+(G_{A_h}))$, i.e. G_A and G_{A_h} are stably isomorphic AF-algebras (see Section 2 for notation). It is verified directly, that $\theta \circ \sigma_{A_h} = \sigma_A \circ \theta$ for $\theta = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$; therefore G_A and G_{A_h} are also strongly stably isomorphic. Notice that the strong stable class of G_A contains more than one representative. Using the Smith normal form of a matrix (see below), one can find that e.g. $Ab_{x-1}(G_A) \cong Ab_{x-1}(G_{A_h}) \cong \mathbb{Z}_{a-1}$, which is in accord with Theorem 1 for $p(x) = x - 1$. However, because the eigenvalues λ_A and $\lambda_{A^2} = \lambda_A^2$ generate the same number field, we have an isomorphism of dimension groups $(K_0(G_A), K_0^+(G_A)) \cong (K_0(G_{A^2}), K_0^+(G_{A^2}))$; on the other hand, because $tr\ (A) \neq tr\ (A^2)$ matrices A and A^2 (and, therefore, the shift automorphisms σ_A and σ_{A^2}) cannot be conjugate. In this case, the proof of Theorem 1 breaks, see Lemma 1 and Section 3; therefore the condition strong stable isomorphism cannot be replaced by the stable isomorphism alone.

3. Proof of Theorem 1

Our proof is based on the following criterion [3, Theorem 6.4]: the dimension groups

$$\mathbb{Z}^n \xrightarrow{\mathrm{A}} \mathbb{Z}^n \xrightarrow{\mathrm{A}} \mathbb{Z}^n \xrightarrow{\mathrm{A}} \cdots \qquad \text{and} \qquad \mathbb{Z}^n \xrightarrow{\mathrm{A}'} \mathbb{Z}^n \xrightarrow{\mathrm{A}'} \mathbb{Z}^n \xrightarrow{\mathrm{A}'} \cdots \tag{4}$$

are order-isomorphic and $\sigma_A, \sigma_{A'}$ are conjugate *if and only if* the matrices A and A' are similar in the group $GL(n,\mathbb{Z})$, i.e. $A' = BAB^{-1}$ for a $B \in GL(n,\mathbb{Z})$. The rest of the proof follows from the structure theorem for the finitely generated modules given by the matrix A over a principal ideal domain [11, p. 43]. The result says the normal form of the module (in our case – over the principal ideal domain $\mathbb{Z}[x]/\langle p(x)\rangle$) is independent of the particular choice of a matrix in the similarity class of A.

Before proceeding to a formal proof, let us give an intuitive idea why $Ab_{p(x)}(G_A)$ is invariant of the similarity class of matrix A. Recall that $\mathbb{Z}[x]/\langle p(x)\rangle$ is isomorphic to the ring of integers O_K of an algebraic number field $K = \mathbb{Q}(\alpha)$, where α is a root of polynomial $p(x)$. Since $p(0) = \pm 1$ one can exclude all rational integer entries of matrix $A \in GL(n,\mathbb{Z})$ using equation $p(\alpha) = 0$; thus one gets $A \in GL(n, O_K)$. But O_K is a principal ideal domain (by hypothesis) and, therefore, one can use the Euclidean algorithm to bring A to a diagonal form (the Smith normal form); the factor of O_K-module $GL(n, O_K)$ by a submodule defined by matrix A is a cyclic abelian group – denoted by $Ab_{p(x)}(G_A)$ – which is independent of the similarity class of matrix A. Let us pass to a step by step argument based on the theory of modules.

Proof. By hypothesis, $\mathbb{Z}[x]/\langle p(x)\rangle$ is a principal ideal domain; we shall consider the following $\mathbb{Z}[x]/\langle p(x)\rangle$-module. If $A \in M_n(\mathbb{Z})$ is an $n \times n$ integer matrix, one endows the abelian group $\mathbb{Z}^n$ with a $\mathbb{Z}[x]/\langle p(x)\rangle$-module structure by defining:

$$p_n(x)v = (p_n(A))v, \quad p_n(x) \in \mathbb{Z}[x]/\langle p(x)\rangle,\ v \in \mathbb{Z}^n. \tag{5}$$

Notice that the obtained module depends on matrix A; we shall write $(\mathbb{Z}^n)^A$ for this module.

Fix a set of generators $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of $(\mathbb{Z}^n)^A$. We shall talk about quotient modules in terms of generators and relations, see e.g. lecture notes by Morandi [6]. The relation submodule can be identified with the kernel of a module homomorphism $\phi_{p(x)} \colon (\mathbb{Z}^n)^A \to \mathbb{Z}^n$ defined by the formula $\{p(x)\varepsilon_1, \ldots, p(x)\varepsilon_n\} \mapsto \sum_{i=1}^n p(x)\varepsilon_i$. The relation matrix is a mapping from the module generators to the relation submodule generators; in our case the relation matrix is $p(A)$. Since the relation submodule depends on the polynomial $p(x)$, the factor-module of $\mathbb{Z}[x]/\langle p(x)\rangle$ modulo ker $\phi_{p(x)}$ will be denoted by $(\mathbb{Z}^n)^A_{p(x)}$.

Let $G = (g_{ij})$ be a matrix over the principal ideal domain [11, p. 43]. It is well- known that by the elementary transformations (the Euclidean algorithm) consisting of (i) an interchange of two rows, (ii) a multiplication of a row by -1, (iii) an addition of a multiple of one row to another and similar operations on columns, brings the matrix (g_{ij}) to a diagonal form:

$$D = \begin{pmatrix} g_1 & & & & & \\ & \ddots & & & & \\ & & g_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \tag{6}$$

where g_i are positive integers, such that $g_i \mid g_{i+1}$; the latter is known as the *Smith normal form* of a matrix over the principal ideal domain [11, p. 44]. The elementary transformations are equivalent to a matrix equation $D = PGQ$, where $P, Q \in GL(n, \mathbb{Z})$.

We claim that matrices $p(A)$ and $p(A')$ have the same Smith normal form. First, notice that $p(A)$ and $p(A')$ are similar matrices. Indeed, we know that A' is a matrix similar to A, i.e. $A' = BAB^{-1}$ for a matrix $B \in GL(n, \mathbb{Z})$; then it is verified directly that $p(A') = Bp(A)B^{-1}$, i.e. $p(A)$ and $p(A')$ are similar matrices. Now let D be the Smith normal form of $p(A)$, then $D = Pp(A)Q$ for some $P, Q \in GL(n, \mathbb{Z})$. If $B \in GL(n, \mathbb{Z})$ is such that $p(A') = Bp(A)B^{-1}$, then PB^{-1} and BQ are also in $GL(n, \mathbb{Z})$. One gets the following identities:

$$PB^{-1}(p(A'))BQ = PB^{-1}(Bp(A)B^{-1})BQ = Pp(A)Q = D. \tag{7}$$

In other words, $p(A')$ has the same Smith normal form as $p(A)$. Recall that the module $(\mathbb{Z}^n)^A_{p(x)}$ can be written as:

$$(\mathbb{Z}^n)^A_{p(x)} \cong \mathbb{Z}_{g_1} \oplus \cdots \oplus \mathbb{Z}_{g_r} \oplus \mathbb{Z}^{n-r}, \tag{8}$$

where $\mathbb{Z}_{g_i} = \mathbb{Z}/g_i\mathbb{Z}$. Since the same set of integers g_i will appear in the diagonal form of the matrix $p(A')$, one gets $Ab_{p(x)}(G_A) \cong Ab_{p(x)}(G_{A'})$ for every choice of the polynomial $p(x)$, such that $p(0) = \pm 1$ and $\mathbb{Z}[x]/\langle p(x)\rangle$ is a principal ideal domain. (In the practical considerations, we often have $r = n$ so that our invariant is a finite abelian group.) Theorem 1 follows now from Corollary 1.

The most important special case of the above invariant is when $p(x) = x-1$ (the Bowen-Franks invariant). The invariant takes the form:

$$Ab_{x-1}(G_A) = \mathbb{Z}^n/(A-I)\mathbb{Z}^n. \tag{9}$$

The Bowen-Franks invariant is covered extensively in the literature [14]; such an invariant has a geometric meaning of tracking an algebraic structure of the periodic points of an automorphism of the lattice $\mathbb{Z}^n$ defined by the matrix A. In particular, the cardinality of the group $Ab_{x-1}(G_A)$ is equal to the total number of the isolated fixed points of the automorphism A. It is easy to see that such a number coincides with $|\det(A-I)|$. $\square$

4. Torsion Conjecture

The basic facts on elliptic curves, complex multiplication, etc., can be found in [12]; an excellent introduction to the subject is [13]. The torsion of rational elliptic curves with complex multiplication was studied in [8]. A link between complex multiplication and G_A was the subject of [7].

4.1. **Teichmüller functor.** Let $\theta \in [0,1)$ be an irrational number. The universal C^*-algebra $\mathcal{A}_\theta$ generated by the unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta}uv$ is called a *noncommutative torus* [9], [3, Chapter 5 (p. 34)], and [10, Exercise 5.8, pp. 86–88]. The torus $\mathcal{A}_\theta$ is not an AF-algebra, but can be embedded into an AF-algebra given by the following Bratteli diagram:

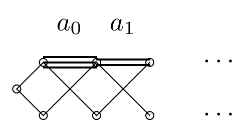

Figure 1. The AF-algebra corresponding to $\mathcal{A}_\theta$.

where $\theta = [a_0, a_1, \dots]$ is the continued fraction of θ [3, p. 65]. A pair of noncommutative tori is said to be stably isomorphic (Morita equivalent) whenever $\mathcal{A}_\theta \otimes \mathcal{K} \cong \mathcal{A}_{\theta'} \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators. The $\mathcal{A}_\theta$ is stably isomorphic to $\mathcal{A}_{\theta'}$ if and only if $\theta' = (a\theta+b)/(c\theta+d)$, where $a,b,c,d \in \mathbb{Z}$ and $ad-bc = 1$. The K-theory of $\mathcal{A}_\theta$ is Bott periodic with $K_0(\mathcal{A}_\theta) = K_1(\mathcal{A}_\theta) \cong \mathbb{Z}^2$. The range of trace on projections of $\mathcal{A}_\theta \otimes \mathcal{K}$ is a subset $\Lambda = \mathbb{Z} + \mathbb{Z}\theta$ of the real line; the set $\Lambda \cong K_0(\mathcal{A}_\theta)$ is known as a pseudo-lattice [5]. The noncommutative torus $\mathcal{A}_\theta$ is said to have *real multiplication*, if θ is a quadratic irrationality; we denote such an algebra by $\mathcal{A}_{RM}$. Real multiplication implies non-trivial endomorphisms of the pseudo-lattice Λ_{RM} given as a multiplication by real numbers – hence the name. Such endomorphisms make a ring under addition and composition of the endomorphisms; the latter is isomorphic to an order of conductor $f \geq 1$ in the ring of integers of quadratic field $\mathbb{Q}(\theta)$. Recall that each order of $\mathbb{Q}(\sqrt{d})$ has the form $\mathbb{Z} + (f\omega)\mathbb{Z}$, where $\omega = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \pmod 4$ and $\omega = \sqrt{d}$ if $d \equiv 2, 3 \pmod 4$. It is known that continued fraction of $\theta = f\omega$ is periodic and has the form $[a_0, \overline{a_1, \ldots, a_n}]$; we shall consider a matrix $A = \prod_{i=1}^{n} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}$.

Lemma 2. $K_0(G_A) \cong K_0(\mathcal{A}_{RM})$.

Proof. It follows easily from the definition of A, that $K_0(G_A) \cong \mathbb{Z} + \mathbb{Z}\theta'$, where $\theta' = \theta - a_0$. In other words, $K_0(G_A) \cong K_0(\mathcal{A}_{RM})$. $\square$

Let $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and for $\tau \in \mathbb{H}$ let $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be a complex torus; we routinely identify the latter with a non-singular elliptic curve via the Weierstrass $\wp$ function [12, pp. 6–7]. Recall that two complex tori are isomorphic, whenever $\tau' = (a\tau + b)/(c\tau + d)$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. If τ is an imaginary quadratic number, elliptic curve is said to have *complex multiplication*; we shall denote such curves by E_{CM}. Complex multiplication means that lattice $L = \mathbb{Z} + \mathbb{Z}\tau$ admits non-trivial endomorphisms given as multiplication of L by certain complex (quadratic) numbers. Again, such endomorphisms make a ring under addition and composition of the endomorphisms; the latter is isomorphic to an order of conductor $f \geq 1$ in the ring of integers of imaginary quadratic field $\mathbb{Q}(\tau)$.

Our calculations of torsion are based on a covariant functor between elliptic curves and noncommutative tori. Roughly speaking, the functor maps isomorphic curves to the stably isomorphic tori; we refer the reader to [7] for the details and terminology. To give an idea, let ϕ be a closed 1-form on a topological torus; the trajectories of ϕ define a measured foliation on the torus. By the Hubbard-Masur Theorem, such a foliation corresponds to a point $\tau \in \mathbb{H}$. The map $F \colon \mathbb{H} \to \partial\mathbb{H}$ is defined by the formula $\tau \mapsto \theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi$, where γ_1 and γ_2 are generators of the first homology of the torus. The following is true: (i) $\mathbb{H} = \partial\mathbb{H} \times (0, \infty)$ is a trivial fiber bundle, whose projection map coincides with F; (ii) F is a functor, which maps isomorphic complex tori to the stably isomorphic noncommutative tori. We shall refer to F as the *Teichmüller functor*. Remarkably, functor F maps E_{CM} to $\mathcal{A}_{RM}$; more specifically, complex multiplication by order of conductor f in imaginary field $\mathbb{Q}(\sqrt{-d})$ goes to real multiplication by an order of conductor f in the real field $\mathbb{Q}(\sqrt{d})$, see an explicit formula for F [7, p. 524].

$-d$	f	$E_{tors}(\mathbb{Q})$, see [Olson 1974] [8] p.196	continued fraction of $\sqrt{f^2d}$	A	$Ab_{x-1}(G_A)$
-2	1	$\mathbb{Z}_2$	$[1,\overline{2}]$	$\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}$	$\mathbb{Z}_2$
-3	1	$\mathbb{Z}_1$ or $\mathbb{Z}_2$	$[1,\overline{1,2}]$	$\begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$	$\mathbb{Z}_2$
-7	1	$\mathbb{Z}_2$	$[2,\overline{1,1,1,4}]$	$\begin{pmatrix}14 & 3\\ 9 & 2\end{pmatrix}$	$\mathbb{Z}_{14}$
-11	1	$\mathbb{Z}_1$	$[3,\overline{3,6}]$	$\begin{pmatrix}19 & 3\\ 6 & 1\end{pmatrix}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$
-19	1	$\mathbb{Z}_1$	$[4,\overline{2,1,3,1,2,8}]$	$\begin{pmatrix}326 & 39\\ 117 & 14\end{pmatrix}$	$\mathbb{Z}_{13} \oplus \mathbb{Z}_{26}$
-43	1	$\mathbb{Z}_1$	$[6,\overline{1,1,3,1,5,1,3,1,1,12}]$	$\begin{pmatrix}6668 & 531\\ 3717 & 296\end{pmatrix}$	$\mathbb{Z}_{59} \oplus \mathbb{Z}_{118}$
-67	1	$\mathbb{Z}_1$	$[8,\overline{5,2,1,1,7,1,1,2,5,16}]$	$\begin{pmatrix}96578 & 5967\\ 17901 & 1106\end{pmatrix}$	$\mathbb{Z}_{221} \oplus \mathbb{Z}_{442}$
-3	2	$\mathbb{Z}_2$ or $\mathbb{Z}_6$	$[3,\overline{2,6}]$	$\begin{pmatrix}13 & 2\\ 6 & 1\end{pmatrix}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
-7	2	$\mathbb{Z}_2$	$[5,\overline{3,2,3,10}]$	$\begin{pmatrix}247 & 24\\ 72 & 7\end{pmatrix}$	$\mathbb{Z}_6 \oplus \mathbb{Z}_{42}$
-3	3	$\mathbb{Z}_1$	$[5,\overline{5,10}]$	$\begin{pmatrix}51 & 5\\ 10 & 1\end{pmatrix}$	$\mathbb{Z}_5 \oplus \mathbb{Z}_{10}$

4.2. **Numerical examples.** We conclude by examples supporting Conjecture 1; they cover all rational E_{CM} [8], except $d = -1$ and $d = -163$.

Remark 3. Note that $E_{tors}(\mathbb{Q}) \subseteq E_{tors}(K)$ since K is a non-trivial extension of $\mathbb{Q}$. The reader can see, that $K = \mathbb{Q}$ only for the first two rows; we do not have specific results for K in other cases, but the table admits existence of such a field. The third column lists all twists of $E(\mathbb{Q})$ satisfying conjecture 1.

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