COMPOSITION OPERATORS ON GENERALIZED WEIGHTED NEVANLINNA CLASS

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Abstract. Let φ be an analytic self-map of open unit disk $\mathbb{D}$. The operator given by $(C_\varphi f)(z) = f(\varphi(z))$, for $z \in \mathbb{D}$ and f analytic on $\mathbb{D}$ is called a composition operator. Let ω be a weight function such that $\omega \in L^1(\mathbb{D}, dA)$. The space we consider is a generalized weighted Nevanlinna class $\mathcal{N}_\omega$, which consists of all analytic functions f on $\mathbb{D}$ such that $\|f\|_\omega = \int_{\mathbb{D}} \log^+(|f(z)|)\omega(z)dA(z)$ is finite; that is, $\mathcal{N}_\omega$ is the space of all analytic functions belong to $L_{\log^+}(\mathbb{D}, \omega dA)$. In this paper we investigate, in terms of function-theoretic, composition operators on the space $\mathcal{N}_\omega$. We give sufficient conditions for the boundedness and compactness of these composition operators.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For any analytic map $\varphi \colon \mathbb{D} \to \mathbb{D}$ we define the composition operator induced by φ, $C_\varphi \colon \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$, as $C_\varphi f = f \circ \varphi$. These operators have been studied on many spaces of analytic functions. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of C_φ in terms of the function-theoretic properties of the induced map φ. We refer to the monographs by Cowen and MacCluer [5], Duren and Schuster [6], Hedenmalm, Korenblum, and Zhu [8], Shapiro [16], and Zhu ([18, 19]) for the overview of the field as of the early 1990's.

Much is known about the structure of C_φ as an operator on classical and weighted Nevanlinna classes associated with the "standard weights"; see, [1, 2, 3, 4, 7, 9, 11, 15], and [17], for example. In this paper, such problems are addressed for the generalized weighted Nevanlinna class with respect to boundedness and compactness. We investigate how the weight function ω and φ determine whether C_φ is bounded or compact.

We consider only radial weights, these arise from the positive and Lebesgue integrable functions $\omega : [0, 1) \to (0, \infty)$, and put $\omega(z) = w(|z|)$ for each $z \in \mathbb{D}$. Let $dA(z) = \pi^{-1} r dr d\theta$ denote the normalized area measure on $\mathbb{D}$

and let ω be the weight function such that $\omega(z)dA(z)$ defines a finite measure on $\mathbb{D}$; that is, $\omega \in L^1(\mathbb{D}, dA)$.

Given such a weight ω we introduce the space $L_{\log^+}(\mathbb{D}, \omega dA)$ of all measurable functions f on $\mathbb{D}$ such that $\|f\| = \int_{\mathbb{D}} \log^+(|f|)\omega dA$ is finite. Since $\log^+ x \leq \log(1+x) \leq \log 2 + \log^+ x$ for $x \geq 0$, a measurable function f on $\mathbb{D}$ belongs to $L_{\log^+}(\mathbb{D}, \omega dA)$ if and only if

$$\|f\|_\omega = \int_{\mathbb{D}} \log(1 + |f(z)|)\omega(z)dA(z) < \infty.$$

The generalized weighted Nevanlinna class is defined as the set $\mathcal{N}_\omega = \mathcal{H} \cap L_{\log^+}(\mathbb{D}, \omega dA)$. If $\omega \equiv 1$, we get the area-Nevalinna class; see [1, 2, 3], and [17]. If $\omega(r) = (1-r)^\alpha$, $\alpha > -1$, we get the weighted Nevanlinna class; see [4, 7], and [11]. Since $p \log^+ x \leq x^p$ for all $p > 0$ and $x \geq 0$, we get that $\mathcal{N}_\omega$ contains the large weighted Bergman spaces A^p_ω which is studied by Kriete and MacCluer [13].

Our main interest is in the space $\mathcal{N}_\omega$ with induced F-norm $\|\cdot\|_\omega$. We will observe later that $\mathcal{N}_\omega$ is an F-space whose topology is stronger than that of uniform convergence. Here an F-space is a complete metrizable topological vector spaces. For the definition and properties of F-space, see [10] and [12].

2. Preliminaries

In this section we prove several auxiliary propositions which will be used in proofs of the main results. We start with the following proposition, which tells us that the point-evaluation $\delta_z : f \mapsto f(z)$ at each $z \in \mathbb{D}$ is bounded linear functional on $\mathcal{N}_\omega$. Moreover, it asserts that sequences that are norm bounded in $\mathcal{N}_\omega$ are uniformly bounded on compact subsets of $\mathbb{D}$.

Proposition 2.1. *Let ω be a weight function and $f \in \mathcal{N}_\omega$. Then*

$$|f(z)| \leq exp\left(\frac{\|f\|_\omega}{2G(1-|z|)}\right), \qquad \textit{for all} \;\; z \in \mathbb{D},$$

where $G(r) = \int_0^r w(\rho)\rho d\rho$.

Proof. If $f \in \mathcal{N}_\omega$, then by the subharmonicity of $\log(1+|f|)$ we get

$$\log(1+|f(z)|) \leq \frac{1}{2\pi}\int_0^{2\pi} \log\left(1 + |f(z+re^{i\theta})|\right) d\theta, \qquad \text{for} \;\; r \in (0, 1-|z|). \tag{1}$$

Now multiply inequaltiy (1) by $\omega(r)rdr$ then integrate the obtained inequality from 0 to ρ, we get

$$\begin{aligned}\int_0^\rho \log(1+|f(z)|)\omega(r)rdr &\leq \frac{1}{2\pi}\int_0^\rho\int_0^{2\pi}\log\left(1+|f(z+re^{i\theta})|\right)d\theta\omega(r)rdr\\ &= \frac{1}{2}\int_{D(\rho,r)}\log\left(1+|f(\zeta)|\right)\omega(\zeta)dA(\zeta)\\ &= \frac{1}{2}\int_{\mathbb{D}}\log\left(1+|f(\zeta)|\right)\omega(\zeta)dA(\zeta)\\ &= \frac{1}{2}\|f\|_\omega,\end{aligned}$$

where $D(\rho,r)=\{z\in\mathbb{D}:|z-\rho|<r\}$ is the Carleson window. Taking $\rho\to 1-|z|$, we get

$$\log(1+|f(z)|)\leq\frac{\|f\|_\omega}{2G(1-|z|)}.$$

Since $\log^+ x\leq\log(1+x)$ for all $x\geq 0$, we get $\log^+(|f(z)|)\leq\dfrac{\|f\|_\omega}{2G(1-|z|)}$. This gives the desired result. □

The following proposition tells us that the generalized weighted Nevanlinna class $\mathcal{N}_\omega$ is an F-space with respect to the F-norm $\|\cdot\|_\omega$.

Proposition 2.2. *$\mathcal{N}_\omega$, with respect to $\|\cdot\|_\omega$, is an F-space.*

Proof. It is easy to check the routine properties of the F-norm, so we just prove completeness. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{N}_\omega$. Then for every $\epsilon>0$ there is $n_0\in\mathbb{N}$ such that

$$\|f_n-f_m\|<\epsilon,\qquad\text{for all}\quad n,m\geq n_0. \tag{2}$$

By proposition 2.1 we have

$$\log(1+|f_n(z)-f_m(z)|)\leq\frac{\|f_n-f_m\|_\omega}{2G(1-|z|)}. \tag{3}$$

From the inequalities (2) and (3), we get $\{f_n\}$ is a sequence in $\mathcal{H}(\mathbb{D})$ and it converges uniformly to some $f\in\mathcal{H}(\mathbb{D})$. By Fatou's Lemma, we can make $\|f_n-f_m\|_\omega$ as small as we wish by choosing m sufficiently large; that is

$$\begin{aligned}\|f-f_m\|_\omega &= \int_{\mathbb{D}}\log(1+|f(z)-f_m(z)|)w(z)dA(z)\\ &\leq \liminf_{n\to\infty}\int_{\mathbb{D}}\log(1+|f_n(z)-f_m(z)|)w(z)dA(z)\\ &< \epsilon,\qquad\text{for}\quad m\geq n_0.\end{aligned}$$

Which gives $f - f_m \in \mathcal{N}_\omega$ and therefore, $f \in \mathcal{N}_\omega$. Hence, the Cauchy sequence $\{f_n\}$ converges to f in $\mathcal{N}_\omega$, which completes the proof. □

The following proposition shows that the topology of $\mathcal{N}_\omega$ is stronger than the uniform convergence, and it can be proved, using an argument similar to the one in the proof of the previous proposition. For convenience, we include the essential parts of the argument.

Proposition 2.3. *The space $\mathcal{N}_\omega$ is an F-space whose topology is stronger than the uniform convergence.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{N}_\omega$. From the previous argument, $\{f_n\}$ converges uniformly to some $f \in \mathcal{H}(\mathbb{D})$. As before, by Proposition 2.1 and Fatou's Lemma, we have $f \in \mathcal{N}_\omega$ and satisfies $\lim_{n\to\infty} \|f - f_n\|_\omega = 0$. Using Proposition 2.1 one more time, we get that $\{f_n\}$ converges uniformly to f in $\mathcal{N}_\omega$. The proof is complete. □

A linear operator is called compact if the image of the unit ball under the operator has compact closure. As usual, compactness of a composition operator C_φ can be characterized as in the next proposition, which is a generalization of Proposition 3.11 in [5] and Proposition 2.3 in [11]. Its proof is just a modification of those in [5] and [11]. For the reader's benefit, we give the proof.

Proposition 2.4. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is compact on $\mathcal{N}_\omega$ if and only if whenever $\{f_n\}$ is bounded in $\mathcal{N}_\omega$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$ then $\lim_{n\to\infty} \|C_\varphi f_n\|_\omega = 0$.*

Proof. Assume that C_φ is compact and suppose $\{f_n\}$ is a bounded sequence in $\mathcal{N}_\omega$ which converges uniformly to 0 on compact subsets of $\mathbb{D}$. Assume, on the contrary, that $\lim_{n\to\infty} \|C_\varphi f_n\|_\omega \neq 0$. Then we can assume $\lim_{n\to\infty} \|g - C_\varphi f_n\|_\omega = 0$ for some $g \in \mathcal{N}_\omega$. Using Proposition 2.1, we have

$$\log(1 + |(g - C_\varphi f_n)(z)|) \leq \frac{\|g - C_\varphi f_n\|_\omega}{2G(1-|z|)}, \qquad \text{for all } z \in \mathbb{D}.$$

Hence, $\{g - C_\varphi f_n\}$ converges uniformly to 0. Since $f_n \to 0$ uniformly and $\{\varphi(z)\}$ is compact set, $f_n(\varphi(z)) \to 0$ for each $z \in \mathbb{D}$. Hence, $g = 0$ and $\lim_{n\to\infty} \|C_\varphi f_n\|_\omega = 0$, which contradicts our assumption.

Conversely, let $\{f_n\}$ be a bounded sequence in $\mathcal{N}_\omega$. It is enough to show that the image under C_φ of the sequence $\{f_n\}$ has a convergent subsequence. By Proposition 2.2, $\{f_n\}$ is bounded uniformly on compact subsets of $\mathbb{D}$ and hence, is a normal family. By Montel's Theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges uniformly on a compact subset of $\mathbb{D}$ to some $f \in$

$\mathcal{H}(\mathbb{D})$. As in the proof of Proposition 2.2, we get $f \in \mathcal{N}_\omega$ and hence, by the hypothesis, we obtain $\lim_{k\to\infty} \|C_\varphi(f - f_{n_k})\|_\omega$, which is what we wanted. $\square$

We find that the following area-formula, see [14], is useful for finding a sufficient condition for the boundedness of C_φ. Here, φ is an analytic self-map of $\mathbb{D}$, and for $\zeta \in \varphi(\mathbb{D})$ let $\{z_j(\zeta)\}$ be a collection of zeros of $\varphi(z) - \zeta$ including multiplicity.

Lemma 2.5. *Let g and Q be non-negative measurable functions on $\mathbb{D}$. Then*

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 Q(z)dA(z) = \int_{\varphi(D)} g(\zeta)\left(\sum_{j\geq 1} Q(z_j(\zeta))\right) dA(\zeta).$$

3. Main Results

In this section we present sufficient conditions for a composition operator C_φ to be bounded and compact on $\mathcal{N}_\omega$. The Littlewood Subordination Principle ([5], Theorem 2.22), which applies to analytic self-maps of $\mathbb{D}$ with $\varphi(0) = 0$, gives the boundedness of C_φ in the following theorem.

Theorem 3.1. *If φ is an analytic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$, then C_φ is a bounded operator on $\mathcal{N}_\omega$ and $\|C_\varphi\| \leq 1$.*

Proof. If $f \in \mathcal{N}_\omega$, then $\log(1 + |f|)$ is subharmonic. Thus, the Littlewood Subordination Principle gives for each $r \in (0, 1)$,

$$\int_0^{2\pi} \log\left(1 + |(f\circ\varphi)(re^{i\theta})|\right)\frac{d\theta}{\pi} \leq \int_0^{2\pi} \log\left(1 + |f(re^{i\theta})|\right)\frac{d\theta}{\pi}.$$

Multiplying the inequality by $\omega(r)rdr$ and integrating with respect to r from 0 to 1, we obtain

$$\begin{aligned}\|C_\varphi f\|_\omega &= \int_0^1\int_0^{2\pi} \log\left(1 + |(f\circ\varphi)(re^{i\theta})|\right)\frac{d\theta}{\pi}\omega(r)rdr \\ &\leq \int_0^1\int_0^{2\pi} \log\left(1 + |f(re^{i\theta})|\right)\frac{d\theta}{\pi}\omega(r)rdr \\ &= \|f\|_\omega,\end{aligned}$$

which gives the boundedness of C_φ and $\|C_\varphi\| \leq 1$ as desired. $\square$

If φ does not fix the origin, replace φ with $\psi\circ\varphi$ where ψ is the Möbius transform of $\mathbb{D}$ which exchanges 0 and $\varphi(0)$. Following a similar argument as that in the proof of Theorem 3.1 and making a natural change of variable, we obtain the known inequality for the standard weight $\omega(r) = (1-r)^{\alpha}$, $\alpha > -1$,

$$\|C_\varphi f\|_\omega \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{2+\alpha} \|f\|_\omega, \qquad \text{for all analytic functions } f \text{ on } \mathbb{D}.$$

Theorem 3.2. *Let φ be an analytic bounded valence self-map of $\mathbb{D}$ and let ω be a weight function such that*

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)}{\omega(|\varphi(z)|)} < \infty.$$

Then the composition operator C_φ is bounded on $\mathcal{N}_\omega$.

Proof. Let $a = \varphi(0)$ and let $\varphi_a(z) = \dfrac{a-z}{1-\bar{a}z}$ be the automorphism of the disk. Consider the function $\psi(z) = \varphi_a \circ \varphi(z)$. It is clear that $\psi : \mathbb{D} \to \mathbb{D}$ is analytic and $\psi(0) = 0$. Since $\varphi_a^{-1} = \varphi_a$, we have $\varphi(z) = \varphi_a \circ \psi(z)$ and $C_\varphi = C_\psi C_{\varphi_a}$. Hence, by Theorem 3.1, for $f \in \mathcal{N}_\omega$ we have

$$\|C_\varphi f\|_\omega = \|C_\psi C_{\varphi_a} f\|_\omega \leq \|C_{\varphi_a} f\|_\omega.$$

On the other hand, it is clear $|\varphi'(z)| \geq \dfrac{1-|a|}{1+|a|}$. Since $\psi(0) = 0$, by the Schwarz Lemma and the hypothesis, there exists an $M > 0$ such that

$$w(z) \leq M\omega(|\varphi_a(z)|), \qquad \text{for all } z \in \mathbb{D}.$$

Thus, by the area-formula, for $f \in \mathcal{N}_\omega$ we obtain

$$\begin{aligned}
\|C_{\varphi_a} f\|_\omega &= \int_{\mathbb{D}} \log(1+|f(\varphi_a(z))|)\omega(z)dA(z) \\
&\leq M\int_{\mathbb{D}} \log(1+|f(\varphi_a(z))|)\omega(|\varphi_a(z)|)dA(z) \\
&\leq M\left(\frac{1+|a|}{1-|a|}\right)\int_{\mathbb{D}} \log(1+|f(\varphi_a(z))|)|\varphi'(z)|\omega(|\varphi_a(z)|)dA(z) \\
&\leq M\left(\frac{1+|a|}{1-|a|}\right)\int_{\varphi(\mathbb{D})} \log(1+|f(\zeta)|)\left(\sum_{j\geq 1}\omega(|\varphi_a(z_j(\zeta))|)\right)dA(\zeta) \\
&\leq kM\left(\frac{1+|a|}{1-|a|}\right)\int_{\varphi(\mathbb{D})} \log(1+|f(\zeta)|)\omega(\zeta)dA(\zeta) \\
&\leq kM\left(\frac{1+|a|}{1-|a|}\right)\|f\|_\omega, \qquad \text{where } k \text{ is the valence of } \varphi.
\end{aligned}$$

This gives the boundedness of C_φ. □

Let us turn to the compactness question. In the next theorem we present a sufficient condition for the composition operator C_φ to be compact.

Theorem 3.3. *Let φ be an analytic self-map of $\mathbb{D}$ with*

$$\limsup_{|z|\to 1} \frac{\omega(z)}{G(1-|\varphi(z)|)} = 0,$$

where $G(r) = \int_0^r \omega(\rho)\rho d\rho$. *Then C_φ is compact on $\mathcal{N}_\omega$.*

Proof. Let $\{f_n\}$ be a bounded sequence in $\mathcal{N}_\omega$ that converges uniformly on compact subsets of $\mathbb{D}$. By Proposition 2.4, it is enough to prove

$$\lim_{n\to\infty} \|C_\varphi f_n\|_\omega = 0.$$

Suppose the hypothesis holds, then for every $\epsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\frac{\omega(z)}{G(1-|\varphi(z)|)} \le \epsilon, \qquad \text{for } \delta < |z| < 1. \tag{4}$$

On the other hand, without loss of generality, we may assume $\varphi(0) = 0$; otherwise we can replace φ with $\psi \circ \varphi$ where ψ is the Möbius transform of $\mathbb{D}$ which exchanges 0 and $\varphi(0)$. Thus, by the Schwarz Lemma, $|\varphi(z)| < |z|$ for all $z \in \mathbb{D}$. Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, there exists an $n_0 \in \mathbb{N}$ such that

$$|f_n(\varphi(z))| < \epsilon, \qquad \text{for all } n > n_0 \text{ and } |z| \le \delta. \tag{5}$$

Using Proposition 2.1 and the inequalities (4) and (5), we have

$$\begin{aligned}
\|C_\varphi f\|_\omega &= \int_{\mathbb{D}} \log(1+|f_n(\varphi(z))|)\omega(z)dA(z)\\
&= \int_{|z|\le\delta} \log(1+|f_n(\varphi(z))|)\omega(z)dA(z)\\
&\quad + \int_{|z|>\delta} \log(1+|f_n(\varphi(z))|)\omega(z)dA(z)\\
&\le \log(1+\epsilon)\int_{|z|\le\delta} \omega(z)dA(z) + \int_{|z|>\delta} \frac{\|f_n\|_\omega}{2G(1-|\varphi(z)|)}\omega(z)dA(z)\\
&\le M\left(\log(1+\epsilon)+2\epsilon\right),
\end{aligned}$$

for some constant $M > 0$. This proves the compactness of C_φ. □

If a "little-oh" condition determines when an operator is compact, it is then relatively straightforward to formulate and prove the corresponding "big-oh" condition that determines when it is bounded. The next theorem gives another sufficient condition for the boundedness of C_φ. A modification of the preceding argument yields the proof of Theorem 3.4, so we omit the proof.

Theorem 3.4. *Let φ be an analytic self-map of $\mathbb{D}$ such that*

$$\limsup_{|z|\to 1} \frac{\omega(z)}{G(1-|\varphi(z)|)} < \infty.$$

Then C_φ is bounded on $\mathcal{N}_\omega$.

References

[1] J. C. Choa and H. O. Kim, *Compact composition operators on the Nevanlinna class*, Pro. Amer. Math. Soc., **125** (1997), 145–151.

[2] J. C. Choa and H. O. Kim, *On function-theoretic conditions characterizing compact composition operators on H^2*, Pro. Japan Acad., **97** (1999), 109–112.

[3] J. C. Choa, H. O. Kim, and J. H. Shapiro, *Compact composition operators on the Smirnov class*, Pro. Amer. Math. Soc., **128** (2000), 2297–2308.

[4] B. R. Choe, H. Koo, and W. Smith, *Carleson measure for the area Nevanlinna spaces and applications*, J. D'Analyse Math., **104** (2008), 207–233.

[5] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC press, Boca Raton, 1995.

[6] P. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, 2004.

[7] A. Haldimann and H. Jarchow, *Nevanlinna algebras*, Studia Math., **147** (2001), 243–268.

[8] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, New York, 2000.

[9] N. Jaoua, *Order bounded composition operators on the Hardy spaces and the Nevanlinna class*, Studia Math., **134** (1999), 35–55.

[10] H. Jarchow, *Locally Convex Spaces*, Teubner-Verlag, Stuttgart, 1981.

[11] H. Jarchow and J. Xiao, *Composition operators between Nevanlinna classes and Bergman spaces with weights*, J. Operator Theory, **46** (2001), 605–618.

[12] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-Spaces Sampler*, London Math. Soc. Lecture Note Ser., Vol. 89, Cambridge Univ. Press, Cambridge, 1984.

[13] T. L. Kriete and B. D. MacCluer, *Composition operators on large weighted Bergman spaces*, Indiana Univ. Math. J., **41** (1992), 755–788.

[14] F. Morgan, *Geometric Measure Theory.* Academic Press, Boston, 1988.

[15] J. H. Shapiro and A. L. Shields, *Unusual topological properties of the Nevanlinna class*, Amer. J. Math, **97** (1976), 915–936.

[16] J. H. Shapiro, *Composition Operators and Classical Function Theory.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[17] J. Xiao, *Compact composition operators on the area-Nevanlinna class*, Exposition. Math., **17** (1999), 255–264.

[18] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.

[19] K. Zhu, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139, Marcel Dekker, Inc., New York, 1990.

MSC2010: 47B38, 47B15, 47B33, 30H05

Key words and phrases: Composition operators, generalized weighted Nevanlinna class, weighted Bergman spaces, compact operators, bounded operators, radial weight function.

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