ON THE MODIFIED FERMAT PROBLEM

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Abstract. For a given positive real number v smaller than $\sqrt{2}$, we consider the Fermat-like configuration consisting of a circle k and a rectangle $ABB'A'$. A point P is on k if and only if the relation $|AD|^2 + |BC|^2 = v^2\,|AB|^2$ holds, where C and D are the intersections of the line AB with the lines $A'P$ and $B'P$, respectively. There are four such rectangles with the side AA' parallel to any given line of symmetry of the circle. This property is shared by all ellipses. When $v = \sqrt{2}$, analogous statements hold for parabolas. Finally, for $v > \sqrt{2}$, this is true for hyperbolas only for its line of symmetry containing the foci. We also show that many geometric properties of this configuration do not depend on a position of a point on the circle. The original Fermat problem corresponds to the case $v = 1$.

1. Introduction – The Fermat Problem

For given different points A and B and any points P_1, P_2, P_3, P_4 in the plane, let $\varphi(P_1P_2, P_3P_4) = \frac{|P_1P_2|^2+|P_3P_4|^2}{|AB|^2}$.

Among the numerous questions that Pierre de Fermat formulated, the following geometric problem is our main concern (see Figure 1).

Fermat Problem . Let P be a point on the semicircle that has the top side AB of the rectangle $ABB'A'$ as a diameter. Let $\frac{|AB|}{|AA'|} = \sqrt{2}$. Let the segments PA' and PB' intersect the side AB in the points C and D, respectively. Then $\varphi(AD, BC) = 1$.

The great Leonard Euler in [3] provided the first rather long proof, which is old fashioned (for his time), and avoids the analytic geometry (which offers rather simple proofs as we shall see later). Several more concise synthetic proofs are now known (see [6], [4, pp. 602, 603], [1, pp. 168, 169] and [5, pp. 181, 264]). A nice description of Euler's proof can be found in [7].

The analytic proofs also reveal that the above relation holds for all points on the circle with the segment AB as a diameter.

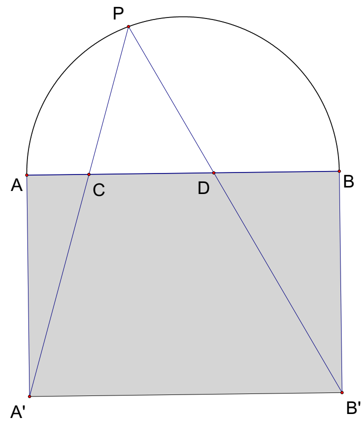

FIGURE 1. The configuration of the Fermat problem.

2. The modified Fermat problem

For a circle, we consider a slightly more general situation when the number 1 in $\varphi(AD, BC) = 1$ is replaced by v^2, where v is a real number satisfying $0 < v < \sqrt{2}$ (see Figure 2).

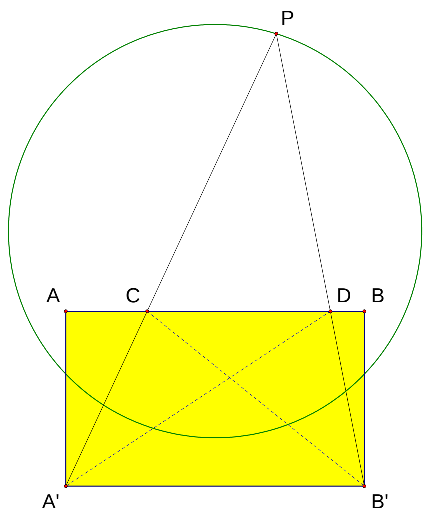

FIGURE 2. The modified Fermat configuration for a circle.

The following result gives this modified form of the Fermat problem.

Theorem 1. *For every positive real number v smaller than $\sqrt{2}$, every circle k and every line π in the plane there are exactly four rectangles $ABB'A'$ such that the lines AA' and π are parallel and the following two statements are equivalent for a point P in the plane:*

(a) *A point P is on the circle k.*

(b) *$\varphi(AD, BC) = v^2$, where C and D are intersections of the line AB with the lines $A'P$ and $B'P$, respectively.*

Proof. We shall use analytic geometry which offers a simple proof. Without loss of generality, we can assume that the line π is the y-axis of the rectangular coordinate system, the center of the circle O is its origin and the equation of the circle is a standard $x^2 + y^2 = r^2$, where r is a positive real number (the radius of the circle).

The coordinates of the points A, B, A' and B' are (a, b), $(a + cd, b)$, $(a, b + c)$ and $(a + cd, b + c)$, respectively, where a, b, c and d are real numbers such that $c \neq 0$ and $d \neq 0$. An arbitrary point P on the plane has coordinates (p, q). From the similar right-angled triangles, we easily find that $C\left(\frac{cp-a(q-b)}{b+c-q}, b\right)$ and $D\left(\frac{cp-(a+cd)(q-b)}{b+c-q}, b\right)$. For an integer n, let $\delta_n = n - v^2$. Note that $\varphi(AD, BC) - v^2$ is $\frac{M_y}{d^2(b+c-q)^2}$, where M_y is the quadratic polynomial

$$d^2\left[c\,\delta_1(2b + c - 2q) + \delta_2(q - b)^2\right] + 2(a - p)(a + cd - p)$$

in the variables p and q.

Now in order that P is on the circle k, the polynomial M_y should be of the form $\lambda(p^2 + q^2 - r^2)$ for some real number $\lambda \neq 0$. This gives the system of six equations in five variables a, b, c, d and λ. Let $L = r\sqrt{2}$ and $K = \sqrt{2\,\delta_2}$. One solution is $a = -\frac{KL}{2\,v}$, $b = \frac{\delta_1 L}{v}$, $c = -\frac{L\delta_2}{v}$, and $d = -\frac{K}{\delta_2}$. Hence, the associated first rectangle $ABB'A'$ has as vertices the pairs $\frac{L}{v}\left(-\frac{K}{2}, \delta_1\right)$, $\frac{L}{v}\left(\frac{K}{2}, \delta_1\right)$, $\frac{L}{v}\left(\frac{K}{2}, -1\right)$, $\frac{L}{v}\left(-\frac{K}{2}, -1\right)$, where we use $\mu(a, b)$ as a shorter notation for the pair $(\mu\,a, \mu\,b)$. The second rectangle is the reflection of the first in the x-axis. We get two additional rectangles by reflecting these rectangles in the y-axis (see Figure 3). The above system of equations has no other solutions.

Finally, if $P(p, q)$ is any point in the plane, then $\varphi(AD, BC) - v^2 = \frac{v^2\delta_2\kappa}{\vartheta^2}$, where $\vartheta = v\,q + L$ and $\kappa = p^2 + q^2 - r^2$. Hence, for the first rectangle $ABB'A'$, the identity $\varphi(AD, BC) = v^2$ holds if and only if the coordinates of the point P satisfy $\kappa = 0$, i.e., if and only if the point P is on the circle k. The same conclusion is true for the remaining three rectangles. □

In the sequel, we shall call any of the above four rectangles $ABB'A'$, the F^v-*rectangle* of the circle k (in the direction π). We shall use the same

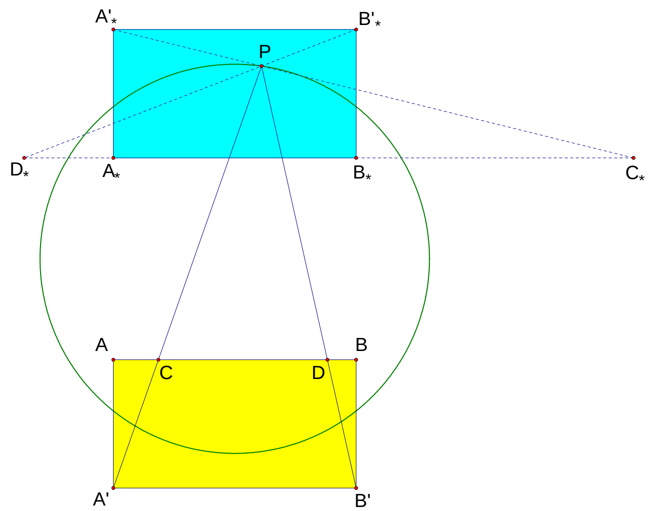

FIGURE 3. The four $F^{\frac{6}{5}}$-rectangles $ABB'A'$, $BAA'B'$, $A_*B_*B'_*A'_*$ and $B_*A_*A'_*B'_*$ in the vertical direction.

name also for other conics. Note that the points C and D (for the first rectangle) are $\frac{L}{2v\vartheta}\,(2v\delta_2 p \mp Kvq \pm KL\delta_1, 2\delta_1\vartheta)$. Throughout the paper, the upper sign goes with the first stated point and the lower sign goes with the second stated point, respectively.

3. The Case of an Ellipse

The following theorem is a version of Theorem 1 for ellipses. Since circles are special ellipses (with eccentricity zero) and their lines of symmetry are all lines through their centers, it follows that Theorem 2 is a generalization of Theorem 1. On the other hand, the ellipse case could also be derived from the circle case by applying an affine stretch.

Theorem 2. *For every positive real number $v < \sqrt{2}$, every ellipse γ and each line of symmetry π of γ there are exactly four rectangles $ABB'A'$ such that the lines AA' and π are parallel and the following two statements are equivalent for a point P in the plane:*

(a) *A point P is on the ellipse γ.*
(b) *$\varphi(AD, BC) = v^2$, where C and D are intersections of the line AB with the lines $A'P$ and $B'P$, respectively.*

Proof. We shall use analytic geometry again and follow the above proof for circles. We first assume that the line π is the y-axis of the rectangular coordinate system, the center of the ellipse O is its origin and the equation of the ellipse is a standard $\frac{x^2}{h^2}+\frac{y^2}{k^2}=1$, where h and k are positive real numbers (the semi-axes of the ellipse).

Once again, the coordinates of the points A, B, A' and B' are (a,b), $(a+cd,b)$, $(a,b+c)$, and $(a+cd,b+c)$, respectively, where a, b, c, and d are real numbers such that $c\neq 0$ and $d\neq 0$. An arbitrary point P on the plane has coordinates (p,q). The coordinates of the intersections C and D, the difference $\varphi(AD,BC)-v^2$ as well as the polynomial M_y have been computed above.

Now in order that P is on the ellipse γ, the polynomial M_y should be of the form $\lambda\left(\frac{p^2}{h^2}+\frac{q^2}{k^2}-1\right)$ for some real number $\lambda\neq 0$. This gives the system of six equations in five variables a, b, c, d, and λ. One solution is $a=-\frac{h\sqrt{\delta_2}}{v}$, $b=\frac{k\sqrt{2}\delta_1}{v}$, $c=-\frac{k\sqrt{2}\delta_2}{v}$, and $d=-\frac{h\sqrt{2}}{k\sqrt{\delta_2}}$ that gives the first rectangle $ABB'A'$ with vertices $\frac{1}{v}\left(-h\sqrt{\delta_2},k\,\delta_1\sqrt{2}\right)$, $\frac{1}{v}\left(h\sqrt{\delta_2},k\,\delta_1\sqrt{2}\right)$, $\frac{1}{v}\left(h\sqrt{\delta_2},-k\,\sqrt{2}\right)$, $\frac{1}{v}\left(-h\sqrt{\delta_2},-k\,\sqrt{2}\right)$. The second rectangle is the reflection of the first in the x-axis. We get two additional rectangles by reflecting these rectangles in the y-axis (see Figure 4). There are no other solutions.

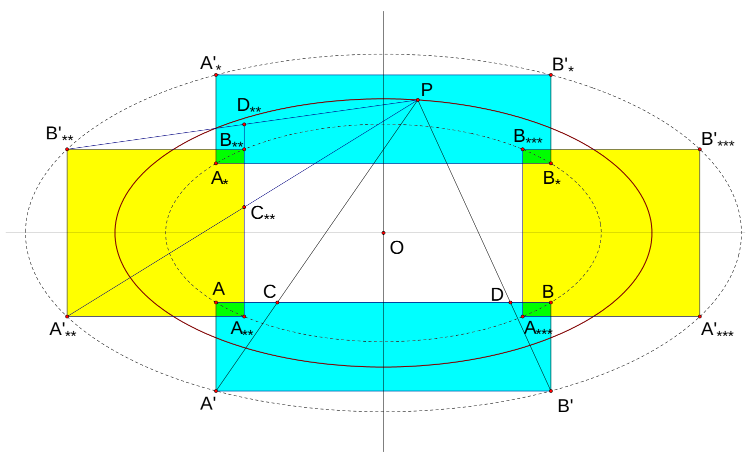

FIGURE 4. The eight $F^{\frac{6}{5}}$-rectangles of the ellipse $x^2+4\,y^2=1$.

By repeating this argument for the x-axis (the second line of symmetry of γ), we shall get analogously four more rectangles. □

Note that these eight F^v-rectangles all have the same area $\frac{2\sqrt{2}h\,k\,\delta_2\sqrt{\delta_2}}{v^2}$ and their vertices lie on two ellipses coaxal with γ (see Figure 4).

4. The Case of a Parabola

When $v = \sqrt{2}$, the Fermat configuration is tied to a parabola. This is explained in the following result.

Theorem 3. *For every parabola δ and every real number $d \neq 0$, there is a rectangle $A_d B_d B'_d A'_d$ such that the side $A_d A'_d$ is parallel to the line of symmetry π of δ and for a point P in the plane the following two statements are equivalent:*

(a) *A point P is on the parabola δ.*
(b) *$\varphi(A_d D_d, B_d C_d) = 2$, where C_d and D_d are intersections of the line $A_d B_d$ with the lines $A'_d P$ and $B'_d P$, respectively.*

Proof. We again assume that the line π is the y-axis of the rectangular coordinate system, the equation of the parabola is a standard $x^2 = 2\,h\,y$ and the coordinates of the vertices A, B, A', and B' of a rectangle $ABB'A'$ with AA' and π parallel are (a, b), $(a + cd, b)$, $(a, b + c)$ and $(a + cd, b + c)$, respectively, where a, b, c, d, and h are real numbers such that $c \neq 0$, $d \neq 0$, and $h > 0$. An arbitrary point P on the plane has coordinates (p, q). The coordinates of the intersections C and D have been computed above. Then $2 - \varphi(AD, BC) = \frac{M_0}{d^2(b+c-q)^2}$, where

$$M_0 = c\,d^2(2\,b + c - 2\,q) + 2(p - a)(a + c\,d - p).$$

Now, in order that P is on the parabola δ, the polynomial M_0 should be of the form $\lambda\left(p^2 - 2\,h\,q\right)$ for some real number $\lambda \neq 0$. This gives the system of five equations in five variables a, b, c, d, and λ. The only solution is $a = \frac{h}{d}$, $b = \frac{3\,h}{2\,d^2}$, $c = -\frac{2\,h}{d^2}$, and $\lambda = -2$ that gives the required rectangles $A_d B_d B'_d A'_d$ with vertices $\frac{h}{d}\left(1, \frac{3}{2\,d}\right)$, $\frac{h}{d}\left(-1, \frac{3}{2\,d}\right)$, $-\frac{h}{d}\left(1, \frac{1}{2\,d}\right)$, $-\frac{h}{d}\left(-1, \frac{1}{2\,d}\right)$, respectively (see Figure 5). $\square$

It is obvious that for every $d \neq 0$ the rectangles $A_d B_d B'_d A'_d$ and $A_{-d} B_{-d} B'_{-d} A'_{-d}$ are symmetric with respect to the line π.

5. The Case of a Hyperbola

Finally, when $v > \sqrt{2}$, the modified Fermat configuration is possible for each hyperbola. This is described in the following theorem.

Theorem 4. *For every hyperbola η and every real number $v > \sqrt{2}$, there are exactly four rectangles $ABB'A'$ such that the side AA' is parallel to the major line of symmetry π of η (going through its foci) such that for a point P in the plane the following two statements are equivalent:*

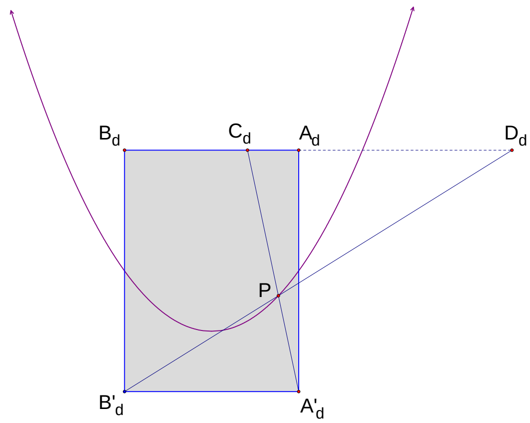

FIGURE 5. The $F^{\sqrt{2}}$-rectangle $A_dB_dB'_dA'_d$ of a parabola.

(a) *A point P is on the hyperbola η.*
(b) $\varphi(AD, BC) = v^2$*, where C and D are intersections of the line AB with the lines $A'P$ and $B'P$, respectively.*

Proof. We assume that the line π is the x-axis of the rectangular coordinate system, the equation of the hyperbola is $\frac{x^2}{h^2} - \frac{y^2}{k^2} = 1$ and the coordinates of the vertices A, B, A', B', and P are (a, b), $(a, b + cd)$, $(a + c, b)$, $(a + c, b + c\,d)$, and (p, q), respectively, where h, k, a, b, c, d, p, and q are real numbers such that $h > 0$, $k > 0$, $c \neq 0$ and $d \neq 0$. The coordinates of the intersections C and D, the difference $\varphi(AD, BC) - v^2$, and the polynomial M_x have been computed above.

The point P is on the hyperbola η if and only if the polynomial M_x has the form $\lambda\left(\frac{p^2}{h^2} - \frac{q^2}{k^2} - 1\right)$ for some real number $\lambda \neq 0$. This gives the system of six equations in five variables a, b, c, d, and λ. One of its four solutions is $a = -\frac{h\,\delta_1\sqrt{2}}{v}$, $b = \frac{k\sqrt{-\delta_2}}{v}$, $c = \frac{h\,\delta_2\sqrt{2}}{v}$, and $d = \frac{k\sqrt{2}}{h\sqrt{-\delta_2}}$ that gives the first rectangle $ABB'A'$ with vertices $\frac{1}{v}\left(-h\delta_1\sqrt{2}, k\sqrt{-\delta_2}\right)$, $-\frac{1}{v}\left(h\delta_1\sqrt{2}, k\sqrt{-\delta_2}\right)$, $\frac{1}{v}\left(h\sqrt{2}, k\sqrt{-\delta_2}\right)$, and $\frac{1}{v}\left(h\sqrt{2}, -k\sqrt{-\delta_2}\right)$. The second rectangle is the reflection of the first in the y-axis. We get two additional rectangles by reflecting these rectangles in the x-axis (see Figure 6). There are no other solutions. Also, when we assume that the line π is the y-axis, repeating the above procedure, we get a system that has no solutions. $\square$

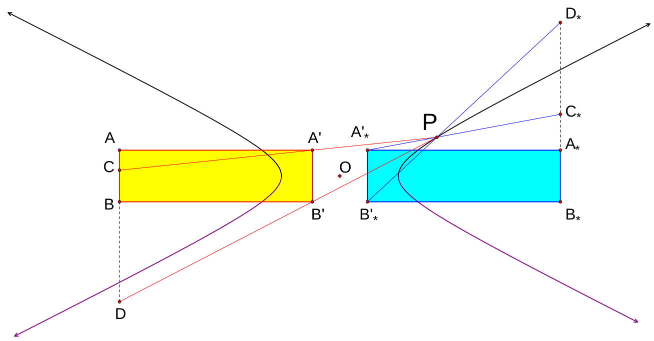

FIGURE 6. The four F^3-rectangles of the hyperbola $x^2 - 4\,y^2 = 1$.

6. INVARIANTS OF THE MODIFIED FERMAT CONFIGURATION

In the rest of this paper, our main goal is to explore what other relationships in the modified Fermat configuration of a circle and its F^v-rectangle $ABB'A'$ remain invariant as the point P changes position on the circle. In other words, we search for statements analogous to (b) that are also equivalent to (a) (both in Theorem 1) and are related to the F^v-rectangle of k. The case $v = 1$ was considered earlier in [2]. Similar results could be also proved for other conics (ellipse, parabola, and hyperbola). These cases have more complicated statements.

Let P' be the reflection of the point P in the x-axis. We remark that most of our results in this section come in related pairs. The second version, which requires no extra proof, comes (for example, already in Theorem 1) by replacing the points C and D with the points C' and D', which are the intersections of the line AB with the lines $P'A'$ and $P'B'$, respectively.

We begin with the diagonals of the trapezium $A'B'DC$ (see Figure 2). It is somewhat unusual that the number v does not appear.

(c) $\varphi(A'D, B'C) = \varphi(A'D', B'C') = 2$.

Proof of (c). With straightforward computations one can easily check that $\varphi(A'D, B'C) - 2 = \frac{\kappa v^2 \delta_2}{\vartheta^2}$. □

We note the following generalization. Let the points A_*, B_*, C_*, D_* satisfy $\mathbf{AA}_* = \lambda\mathbf{AA}'$, $\mathbf{BB}_* = \lambda\mathbf{BB}'$, $\mathbf{BC}_* = \mu\mathbf{BC}$, $\mathbf{AD}_* = \mu\mathbf{AD}$ for real numbers λ and μ. Then $\varphi(A_*D_*, B_*C_*) = \delta_2\lambda^2 + v^2\mu^2$.

For points X and Y, let $X \oplus Y$ be the center of the square built on the segment XY such that the triangle $X(X \oplus Y)Y$ has the positive orientation (counterclockwise). When the point $X \oplus Y$ is shortened to M, then M^* denotes $Y \oplus X$.

The midpoints G, H, G', H' of the segments AC, BD, AC', BD' and the top N of the semicircle over AB are used in the next two statements. In other words, $N = B \oplus A$. The midpoint $M\left(0, \frac{\delta_1 L}{v}\right)$ of the segment AB appears in the statement (e) (see Figure 7).

(d) $\varphi(NG, NH) = \varphi(NG', NH') = \frac{2+v^2}{4}$.

(e) $\varphi(MG, MH) = \varphi(MG', MH') = \frac{v^2}{4}$.

Proof of (d) and (e). This time the differences $\varphi(NG, NH) - \frac{2+v^2}{4}$ and $\varphi(MG, MH) - \frac{v^2}{4}$ both simplify to the following quotient $\frac{v^2 \delta_2 \kappa}{4\,\vartheta^2}$, which has the factor κ again. $\square$

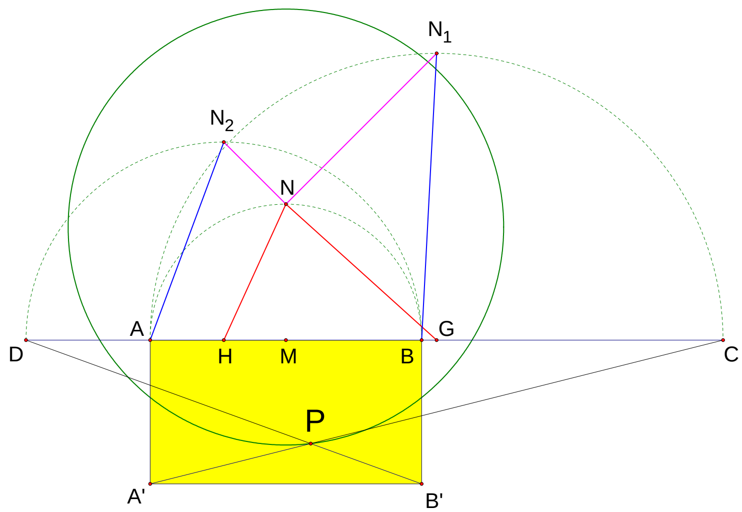

FIGURE 7. The quotients $\varphi(NG, NH)$, $\varphi(AN_2, BN_1)$ and $\varphi(NN_1, NN_2)$ are equal to $\frac{1}{2} + \frac{v^2}{4}$, $1 + \frac{v^2}{2}$, and $\frac{v^2}{2}$.

Let G_s, H_s, G'_s, H'_s be the points that divide the segments NG, NH, NG', NH' in the same ratio $s \neq -1$ (i.e., $NG_s : G_sG = s : 1$, etc.).

(f) $\varphi(MG_s, MH_s) = \varphi(MG'_s, MH'_s) = \frac{s^2 v^2 + 2}{4(s+1)^2}$.

(g) $\varphi(NG_s, NH_s) = \varphi(NG'_s, NH'_s) = \frac{(v^2+2)\, s^2}{4(s+1)^2}$.

Proof of (f). Since $\frac{L}{2(s+1)}\left(\frac{s(2\delta_2 p \mp 2Kq \mp KLv)}{2\vartheta}, \frac{K + 2\delta_1(s+1)}{v}\right)$ are the coordinates of G_s and H_s, the difference $\varphi(MG_s, MH_s) - \frac{s^2v^2+2}{4(s+1)^2}$ is $\frac{s^2 v^2 \delta_2 \kappa}{4(s+1)^2\,\vartheta^2}$. $\square$

Let N_1, N_2, N_3, N_4 denote the highest points on the semicircles built on the segments AC, BD, AC', BD' above the line AB. In other words, $N_1 = C \oplus A$, $N_2 = B \oplus D$, $N_3 = C' \oplus A$, $N_4 = B \oplus D'$ (see Figure 7).

(h) $\varphi(BN_1, AN_2) = \varphi(BN_3, AN_4) = 1 + \frac{v^2}{2}$.
(i) $\varphi(NN_1, NN_2) = \varphi(NN_3, NN_4) = \frac{v^2}{2}$.

Proof of (h) and (i). Since N_1 and N_2 have the coordinates $\frac{L}{4\vartheta}\left(n_{\mp}, \frac{m_{\mp}}{v}\right)$, with $n_{\mp} = 2\delta_2 p \mp 2Kq \mp vKL$ and $m_{\mp} = 2v(\delta_1 q \pm \delta_2 p) + L(K\delta_2 + 4\delta_1)$, we get $\varphi(BN_1, AN_2) - \left(1 + \frac{v^2}{2}\right) = \varphi(NN_1, NN_2) - \frac{v^2}{2} = \frac{v^2\delta_2\kappa}{2\,\vartheta^2}$. □

The following statements also use the points N_1, N_2, N_3 and N_4. However, they do not use the function φ.

(j) $\frac{|N_1N_2|}{|AN|} = v$ and $\frac{|N_3N_4|}{|AN|} = v$.
(k) $|N_1N_2| = |N_3N_4|$.
(l) $|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 = 2\,v^2\,|AB|^2$.

Proof of (j) and (l). We easily get $|N_1N_2|^2 - v^2|AN|^2 = \frac{L^2\,\delta_2^2\,\kappa}{\vartheta^2}$.

Since N_3 and N_4 have the coordinates $\frac{L}{4\eta}\left(h_{\mp}, \frac{k_{\mp}}{v}\right)$, with $\eta = vq - L$, $h_{\mp} = -2\delta_2 p \mp 2Kq \pm vKL$ and $k_{\mp} = 2v(2\,\delta_1 q \mp \delta_2 p) - L(K\delta_2 + 4\delta_1)$, the sum $|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 - 2\,v^2\,|AB|^2$ is equal to

$$\frac{8\,r^2\,\delta_2^2(v^2\,q^2 + 2\,r^2)\kappa}{\eta^2\,\vartheta^2}.$$

□

Note that $|N_1N_2^*|^2 + |N_2^*N_3^*|^2 + |N_3^*N_4|^2 + |N_4N_1|^2 = 2\,v^2\,|AB|^2$ if and only if the point P is on the ellipse $\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1$, where $a = r\sqrt{3 - \frac{2}{v^2}}$ and $b = r\sqrt{\frac{3}{2} - \frac{1}{v^2}} = \frac{a}{\sqrt{2}}$.

Let $m = \frac{v^2}{2}$, $n = \frac{1+m}{2}$, $o = 1 + m$, $t = (s+1)^2$, and $p = n + \frac{n}{t}$. Let $N_5 = A \oplus D$, $N_6 = C \oplus B$, $N_7 = A \oplus D'$, and $N_8 = C' \oplus B$.

(m) $\varphi(AN_5, BN_6) = \varphi(AN_7, BN_8) = m$.
(n) $\varphi(GN_6, HN_5) = \varphi(G'N_8, H'N_7) = n$.
(o) $\varphi(NN_5, NN_6) = \varphi(NN_7, NN_8) = o$ and $\varphi(G_sN_6, H_sN_5) = p$.

Proof of (m). Since N_5 and N_6 have the coordinates $\frac{L}{4\,v\,\vartheta}(s_{\mp}, t_{\mp})$, where $s_{\mp} = \delta_2(2\,v\,p \mp KL)$ and $t_{\mp} = 2\,v[(2\delta_1 - K)q \mp \delta_2 p] + L(4\delta_1 - K\,v^2)$, we get $\varphi(AN_5, BN_6) - \frac{v^2}{2} = \frac{\delta_2 v^2\kappa}{2\,\vartheta^2}$. □

The next six statements use the centers of squares on the segments CD and $C'D'$. Let $M_1 = C \oplus D$ and $M_2 = C' \oplus D'$.

(p) $|N\,M_1| = v\,|A\,N|$ and $|N\,M_2| = v\,|A\,N|$.

(q) $|N\,M_1| = |N\,M_2|$.

Proof of (p) and (q). Since $M_1 = \frac{L}{2\,v\,\vartheta}\,(2\,v\delta_2 p, v(2\delta_1 - K)q + L\delta_1(K+2))$, the difference $|M_1N|^2 - v^2\,|AN|^2$ is equal to $\frac{2\,r^2\,\delta_2^2\kappa}{\vartheta^2}$. In a similar way, we get $|M_2N|^2 - |M_1N|^2 = \frac{4\,L^3\,\delta_2^2\,q\,v\,\kappa}{\eta^2\vartheta^2}$. □

(r) $\varphi(M_1N_1, M_1N_2) = \varphi(M_2N_3, M_2N_4) = \frac{v^2}{2}$.

For any point X in the plane, let G_1, G_2, G_3, G_4, G_5, and G_6 denote the centroids of the triangles ACX, CDX, DBX, $AC'X$, $C'D'X$, and $BD'X$, respectively.

(s) $\varphi(G_2G_1, G_2G_3) = \varphi(G_5G_4, G_5G_6) = \frac{v^2}{9}$.

(t) $|G_2G_1|^2 + |G_2G_3|^2 = |G_5G_4|^2 + |G_5G_6|^2$.

Proof of (s). If $X = (x, y)$, then the points G_1, G_2, and G_3 have the same ordinate $\frac{y}{3} + \frac{2L\delta_1}{3\,v}$ while their abscissae are $\frac{x}{3} - \frac{L(2Kq - 2\delta_2 p + v\,K\,L)}{6\,\vartheta}$, $\frac{x}{3} + \frac{2\,L\,\delta_2 p}{3\,\vartheta}$, and $\frac{x}{3} + \frac{L(2Kq + 2\delta_2 p + v\,K\,L)}{6\,\vartheta}$, respectively. Now we can easily get that the difference $\varphi(G_2G_1, G_2G_3) - \frac{v^2}{9}$ is $\frac{v^2\delta_2\kappa}{9\,\vartheta^2}$. □

We use U and V to denote the midpoints of the segments CC' and DD', respectively (see Figure 8).

(u) $\varphi(NU, NV) = 1$.

(v) $\varphi(MU, MV) = \frac{1}{2}$.

(w) $\varphi(N_6U, N_5V) = \frac{v^2}{2}$.

Proof of (u) and (v). Since $\frac{L}{2\,v}\left(-\frac{\delta_2 L(2\,v\,p \pm K L)}{\vartheta\,\eta} \mp K, 2\delta_1\right)$ are U and V, respectively, we get $\varphi(NU, NV) - 1 = \varphi(MU, MV) - \frac{1}{2} = \frac{\delta_2 L^2\,v^2\kappa}{\eta^2\vartheta^2}$. □

Let $W = U \oplus V$ (see Figure 8).

(x) $\varphi(AW, BW) = 1$ and $\varphi(NW, NW^*) = 1$.

(y) $\varphi(WN_i, WN_j) = \frac{v^2}{2}$, for $i \in \{1, 3\}$ and $j \in \{2, 4\}$.

(z) The center W lies on the circle that has the segment AB as a diameter.

Proof of (z). Since $W = \frac{-L}{2\,v\vartheta\eta}\left(2\delta_2 v\,L\,p, (K - 2\delta_1)v^2\,q^2 + \delta_1 L^2(K+2)\right)$, we get that $|WM|^2 - \frac{|AB|^2}{4}$ equals $\frac{4\,r^2\,\delta_2^2\,\kappa}{\eta^2\,\vartheta^2}$. □

(a1) The lines WN_1 and WN_2 are perpendicular.

(b1) The lines WN_3 and WN_4 are perpendicular.

Proof of (a1). The lines WN_1 and WN_2 have equations $a(x, y) = \lambda$ and $b(x, y) = \mu$, respectively, where $a = a(x, y)$ and $b = b(x, y)$ are homogenous linear functions and λ and μ are real numbers. Let $S = x + y$ and

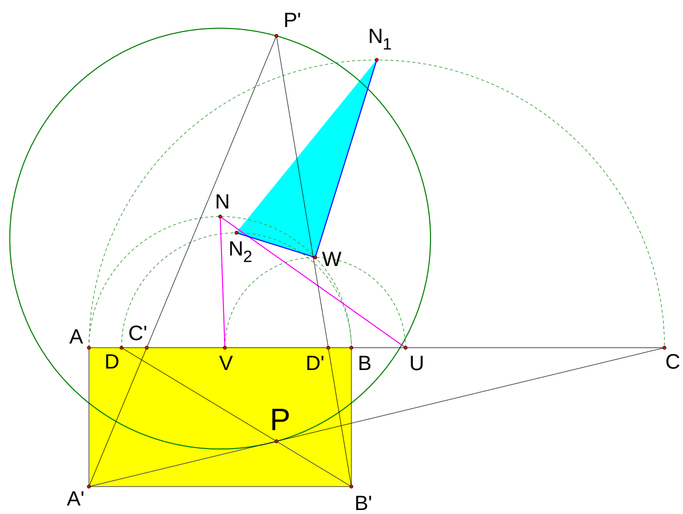

FIGURE 8. $\varphi(NU, NV) = 1$ and $\varphi(WN_1, WN_2) = \frac{v^2}{2}$.

$R = x - y$. Then $a = [Kv(L^2 - 2\,q^2) + 2\delta_2 L\,p]S - \delta_2 q(2\,p\,v + KL)R$. Similarly, we have $b = [Kv(L^2 - 2\,q^2) - 2\delta_2 L\,p]R + \delta_2 q(2\,p\,v - KL)S$. These lines are perpendicular if and only if $\frac{L^3 \delta_2^2 q\,v\,\kappa}{\vartheta^2\,\eta^2}$ is zero. $\square$

Let $K_1 = B \oplus N_1$, $K_2 = N_2 \oplus A$, $K_3 = B \oplus N_3$, $K_4 = N_4 \oplus A$. These points can be defined more simply. They all are at the same height as N and vertically above the points N_6, N_5, N_8, N_7, respectively. Let L_1, L_2, L_3, L_4 be the reflections of the points K_1, K_2, K_3, K_4 in the line AB, respectively. The next four statements use rather exotic numbers.

(c1) $\varphi(A'K_2, B'K_1) = \varphi(A'K_4, B'K_3) = \frac{5}{2} - \frac{3\,v^2}{4} + K$.
(d1) $\varphi(A'L_2, B'L_1) = \varphi(A'L_4, B'L_3) = \frac{5}{2} - \frac{3\,v^2}{4} - K$.

Proof of (c1). Since $\frac{L}{4\,v\,\vartheta}\,(\delta_2(2\,v\,p \pm KL), 2\vartheta(K + 2\delta_1))$ are the coordinates of K_1 and K_2, we get that $\varphi(A'K_2, B'K_1) - \left(\frac{5}{2} - \frac{3\,v^2}{4} + K\right)$ is $\frac{v^2 \delta_2\,\kappa}{4\,\vartheta^2}$. $\square$

Replacing A' and B' with A and B in (c1) and (d1) we get the number $\frac{1}{2} + \frac{v^2}{4}$ as the common value of the function φ in all four cases.

Let S_1 and T_1 denote the midpoints of the segments $A'C$ and $B'D$, respectively. Similarly, let S_2 and T_2 be the midpoints of the segments

$A'C'$ and $B'D'$, respectively. Note that

$$\varphi(G_s S_1, H_s T_1) = \varphi(G'_s S_2, H'_s T_2) = \tfrac{(K\,s+2)^2+2\,s\,K^2+4(K+1)}{8(s+1)^2}.$$

(e1) $\varphi(NS_1, NT_1) = \varphi(NS_2, NT_2) = 1 + \frac{K}{2}$.
(f1) $\varphi(MS_1, MT_1) = \varphi(MS_2, MT_2) = \frac{1}{2}$.

Proof of (f1). From the right-angled triangles MGS_1 and MHT_1 and (e), we get that the sum

$$|MS_1|^2 + |MT_1|^2 = (|MG|^2 + |GS_1|^2) + (|MH|^2 + |HT_1|^2)$$

is $(|MG|^2 + |MH|^2) + \frac{|A'A|^2}{2} = \frac{v^2}{4}\,|AB|^2 + \left(\frac{1}{2} - \frac{v^2}{4}\right)|AB|^2 = \frac{1}{2}\,|AB|^2$. □

By replacing the point N with its reflection N^* in (e1) on the right hand side the $+$ sign changes into the opposite sign $-$.

For points X and Y, let ϱ_X^Y be the reflection of the point X in the point Y. Let $Q = \varrho_A^D$, $R = \varrho_B^C$, $Q' = \varrho_A^{D'}$, $R' = \varrho_B^{C'}$.

(g1) $\varphi(A'Q, B'R) = \varphi(A'Q', B'R') = 3\,v^2 + 2$.

Proof of (g1). Since $\frac{L}{2\,v\,\vartheta}\,(4v\delta_2 p \pm 3Kvq \pm KL(1-2\delta_1), 2\delta_1\vartheta)$ are the coordinates of Q and R, we get that $\varphi(A'Q, B'R) - (3\,v^2+2)$ is equal to $\frac{4\,v^2\delta_2\,\kappa}{\vartheta^2}$. □

We conclude with the following three additional invariant properties of the points Q, R, Q', and R' that could be established by the same method.

(h1) $\varphi(AQ, BR) = \varphi(AQ', BR') = 4\,v^2$.
(i1) $\varphi(N_5Q, N_6R) = \varphi(N_7Q', N_8R') = \frac{5\,v^2}{2}$.
(j1) $\varphi(K_2Q, K_1R) = \varphi(K_4Q', K_3R') = \frac{9\,v^2}{4} + \frac{1}{2}$.

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