ON WILLIAMS NUMBERS WITH THREE PRIME FACTORS

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Abstract. Let $a \in \mathbb{Z} \setminus \{0\}$. A positive squarefree integer N is said to be an *a-Korselt number* (*K_a-number*, for short) if $N \neq a$ and $p - a$ divides $N - a$ for each prime divisor p of N. By an *a-Williams number* (*W_a-number*, for short) we mean a positive integer which is both an *a-Korselt number* and $(-a)$-*Korselt number*.

This paper proves that for each a there are only finitely many *W_a-numbers* with exactly three prime factors, as conjectured in 2010 by Bouallegue-Echi-Pinch.

1. Introduction

We start by defining Korselt numbers.

Definition 1.1. *Let $a \in \mathbb{Z} \setminus \{0\}$. A positive squarefree integer N is said to be an a-*Korselt number *(K_a-number, for short) if $N \neq a$ and $p - a$ divides $N - a$ for each prime divisor p of N.*

For example, 6 is a 4-Korselt number and $231 = 3*7*11$ is a (-9)-Korselt number.

It's clear from the definition that if N is an *a-Korselt number*, then a cannot be equal to any p dividing N.

Korselt numbers were introduced by Echi [3] as a natural generalization of Carmichael numbers which are exactly K_1-numbers and characterized by Korselt by the following criterion.

Korselt's criterion ([6], [2, p. 133]): *A composite odd number n is a Carmichael number if and only if n is squarefree and $p - 1$ divides $n - 1$ for every prime p dividing n.*

Korselt numbers were then further investigated in [1], [3] and [4]. In [7], Williams investigated Carmichael numbers N such that $p+1$ divides $N+1$ for each prime p dividing N. This motivates Echi [3] to introduce the following type of numbers.

Definition 1.2. *Let $a \in \mathbb{Z} \setminus \{0\}$ and N be a squarefree composite number. We say that N is an a-*Williams number *(W_a-number, for short) if N is both an a-Korselt number and $(-a)$-Korselt number.*

For example, $231 = 3 * 7 * 11$ is a 9-Williams number.

It is not known whether there are $k \geq 3$ and a such that there are infinitely many K_a-numbers with k prime factors, or such that there are infinitely many W_a-numbers with k prime factors.

Echi conjectured in [1] and [3] that for each a, there exist infinitely many K_a-numbers with $k \geq 3$ prime factors and proved that for each a, there exist only finitely many K_a-numbers with exactly two prime factors.

Also, Echi conjectured in [1] that there exist only finitely many W_a-numbers with $k \geq 3$ prime factors. More precisely, Echi claims that N is a W_a-number with k prime factors if and only if $k = 3$, $a = 3p$, and $N = p(3p-2)(3p+2)$ where p, $(3p-2)$, and $(3p+2)$ are all primes.

In this paper we prove that for each a there are only finitely many W_a-numbers with three prime factors.

We start by fixing some notations for the following sections. Let $a \in \mathbb{Z} - \{0\}$ and $1 = p_0 < p_1 < \cdots < p_d$, $d \geq 2$ such that the p_i's are primes for each $i \in \{1, 2, \ldots, d\}$ and $N = p_1 p_2 \cdots p_d$ is a K_a-number. Set $q = p_{d-1}$, $r = p_d$, and

$$P = \begin{cases} \prod_{i=1}^{d-2} p_i, & \text{if} \quad d \geq 3; \\ 1, & \text{if} \quad d = 2. \end{cases}$$

Our overall strategy is to derive upper bounds for N in terms of a and P. In this study several cases are discussed and the cases $a < 0$ and $a > 0$ are handled separately.

2. Some Properties Of K_a-Numbers

In this section, we prove some relations between the divisors of N and we establish some inequalities which are useful in Section 3.

We suppose in this section that $\gcd(a, q) = \gcd(a, r) = 1$.

Proposition 2.1. *There exist integers α and β such that*

$$\begin{cases} Pq - 1 = \alpha(r - a); \\ Pr - 1 = \beta(q - a). \end{cases}$$

Proof. As N is a K_a-number, then for each i, $p_i - a$ divides $N - a = N - p_i + p_i - a = p_i(\frac{N}{p_i} - 1) + p_i - a$. If $\gcd(p_i, p_i - a) = 1$, then N is a K_a-number is equivalent to $p_i - a$ divides $\frac{N}{p_i} - 1$.

Since $\gcd(q,a) = \gcd(r,a) = 1$, $q-a$ divides $Pr-1$ and $r-a$ divides $Pq-1$. Therefore, there exist a nonzero integers α and β such that

$$\begin{cases} Pq-1 = \alpha(r-a); \\ Pr-1 = \beta(q-a). \end{cases} \qquad (F_1)$$

□

Let $\Delta = \alpha\beta - P^2$. Then we have the following proposition.

Proposition 2.2.

(1) $\Delta = 0$ *if and only if* $P = 1$, $\alpha = \beta = -1$, *and so* $q + r = a + 1$.

(2) *If* $\Delta \neq 0$, *then*

$$q = \frac{(aP-1)(P+\alpha)}{\Delta} + a$$

and

$$r = \frac{(aP-1)(P+\beta)}{\Delta} + a.$$

Proof.

(1) If $\Delta = 0$, then $P^2 = \alpha\beta$ and by (F_1) we obtain

$$(Pq-1)(Pr-1) = \alpha\beta(r-a)(q-a) = P^2(r-a)(q-a).$$

Hence, P divides 1 and so $P = 1$. Therefore, as $\alpha\beta = P^2 = 1$ we obtain either $\alpha = \beta = 1$ or $\alpha = \beta = -1$.

We claim that $\alpha = \beta = -1$; indeed, if $\alpha = \beta = 1$ and as $P = 1$, then by (F_1), we obtain $q-r = (q-1)-(r-1) = (r-a)-(q-a) = r-q$, which implies that $q = r$, a contradiction.

So by (F_1), we obtain $a = q+r-1$. The converse is obvious.

(2) By (F_1) we obtain

$$\begin{cases} r = \dfrac{Pq-1+\alpha a}{\alpha}; \\ \\ q = \dfrac{Pr-1+\beta a}{\beta}. \end{cases}$$

Substituting r in the expression of q, we get

$$q = \frac{P.\dfrac{Pq-1+\alpha a}{\alpha} - 1 + \beta a}{\beta} = \frac{P^2 q - P + \alpha a P - \alpha + \beta\alpha a}{\alpha\beta}.$$

This implies that

$$q(P^2 - \alpha\beta) + \alpha\beta a + \alpha(aP-1) - P = 0. \qquad (F_2)$$

Similarly, we prove that

$$r(P^2 - \alpha\beta) + \alpha\beta a + \beta(aP-1) - P = 0. \qquad (F_3)$$

Now, suppose that $\Delta \neq 0$. Then by (F_2) we get

$$q = \frac{\alpha\beta a + \alpha(aP-1) - P}{\alpha\beta - P^2} = \frac{(aP-1)(P+\alpha)}{\Delta} + a.$$

Similarly by (F_3), we obtain

$$r = \frac{(aP-1)(P+\beta)}{\Delta} + a.$$

□

We maintain throughout the rest the same definitions for Δ, α, and β as in Proposition 2.1.

Theorem 2.3 ([1])**.** *Let $a \in \mathbb{Z} \setminus \{0\}$. Then the following properties hold.*

(1) *If $a \leq 1$, then each composite squarefree K_a-number has at least three prime factors.*
(2) *Suppose that $a > 1$. Let $q_1 < q_2$ be two prime numbers and $N := q_1 q_2$. If N is an a-Korselt number, then $q_1 < q_2 \leq 4a - 3$. In particular, there are only finitely many a-Korselt numbers with exactly two prime factors.*

Proposition 2.4. *Let N be a K_a-number such that $a < 0$. Then we have*

(1) $0 < \alpha < \beta$ *and* $\alpha < P$.
(2) $\Delta < 0$ *and for* $d = 3, \mid \Delta \mid < 2 \mid aP - 1 \mid$.
(3) $d \geq 3$ *and* $\max\left(1, \dfrac{P^2(p_{d-2} + 2 + a) - 2P}{p_{d-2} + 2 - a}\right) < \alpha\beta < P^2$.

Proof.

(1) As $a < 0$ and

$$\begin{cases} Pq - 1 &= \alpha(r - a); \\ Pr - 1 &= \beta(q - a). \end{cases}$$

we get $\alpha > 0$ and $\beta > 0$, since $q < r$ then $Pq - 1 < Pr - 1$. Therefore, $\alpha(r - a) < \beta(q - a)$. Hence, $\alpha < \beta\left(\dfrac{q-a}{r-a}\right) < \beta$ and so $0 < \alpha < \beta$.

Suppose that $P \leq \alpha$. Then $Pq - 1 = \alpha(r - a) \geq P(r - a)$. This implies that $Pq > P(r-a)$ and so $q > r - a > r$, a contradiction with $q < r$. Thus, $\alpha < P$.

(2) By Proposition 2.2, we have $\Delta \neq 0$.

Suppose that $\Delta > 0$. As in addition $q = \dfrac{(aP-1)(P+\alpha)}{\Delta} + a$, $\alpha > 0$ and $a < 0$ we obtain $q < 0$, which is not possible. Then $\Delta < 0$.

Now, suppose that $d = 3$. Then P is prime such that $P < q$ and we have

$$\mid \Delta \mid = \frac{\mid aP - 1 \mid (P + \alpha)}{q - a} < \frac{\mid aP - 1 \mid (P + \alpha)}{P}.$$

As $0 < \alpha < P$, then

$$| \Delta | < \frac{| aP - 1 | 2P}{P} = 2 | aP - 1 | .$$

(3) By Theorem 2.3, we immediately obtain $d \geq 3$.

We claim that $q - p_{d-2} \geq 2$. Indeed if this is not the case, then $q - p_{d-2} = 1$. This is equivalent to ($q = 2$ and $p_{d-2} = 1$) or ($q = 3$ and $p_{d-2} = 2$). But as $d \geq 3$ we have p_{d-2} is prime, then the first case (i.e., $p_{d-2} = 1$) is not possible.

Suppose that ($q = 3$ and $p_{d-2} = 2$) then $N = 6r$ (i.e., $q = 3 < r$) is a K_a-number, so by (F_1) we have $r - a$ divides $6 - 1 = 5$. This implies that $r - a \leq 5$, but as $a \leq -1$ and $r \geq 5$, we get $r - a \geq 6$, a contradiction.

Now, as $p_{d-2} + 2 \leq q$ and $q - a = \dfrac{(aP-1)(P+\alpha)}{\Delta}$, we obtain

$$0 < p_{d-2} + 2 - a \leq \frac{(aP-1)(P+\alpha)}{\Delta} = \frac{| aP - 1 | (P+\alpha)}{| \Delta |}.$$

This implies that

$$| \Delta | \leq \frac{| aP - 1 | (P+\alpha)}{p_{d-2} + 2 - a} \leq \frac{2P(-aP+1)}{p_{d-2} + 2 - a}.$$

Hence, as $| \Delta | = P^2 - \alpha\beta$, we obtain

$$\alpha\beta \geq P^2 + \frac{2P(aP-1)}{p_{d-2} + 2 - a} = \frac{P^2(p_{d-2} + 2 + a) - 2P}{p_{d-2} + 2 - a}.$$

So, we conclude that

$$\max\left(1, \frac{P^2(p_{d-2} + 2 + a) - 2P}{p_{d-2} + 2 - a}\right) < \alpha\beta < P^2.$$

□

Now we will give a similar result for $a > 0$ as given for $a < 0$ in Proposition 2.4.

Proposition 2.5. *Let N be a K_a-number such that $N = Pqr$ and $0 < a < q < r$. Then we have*

(1) $\Delta > 0$ *and* $0 < \alpha < \beta$.

(2) $0 < \alpha < \dfrac{a+1}{2}P$.

(3) *i) If* $a \geq p_{d-2} + 2$ *then* $P^2 < \alpha\beta < \dfrac{(a+1)(a+2)}{2}P^2$.

ii) If $a < p_{d-2} + 2$ *then* $P^2 < \alpha\beta < \dfrac{P^2(2p_{d-2} + 4 + a(a+1))}{2(p_{d-2} + 2 - a)}$.

Proof. (1) By (F_1), we have $0 < \alpha < \beta$.

It is clear that $\Delta = \dfrac{(aP-1)(P+\alpha)}{q-a}$ is a positive integer.

(2) As $Pq - 1 = \alpha(r-a)$ and $r \geq q+1$, then

$$\alpha = \frac{Pq-1}{r-a} \leq \frac{Pq-1}{q+1-a}.$$

Define for a fixed integer $a > 0$ the function

$$f\colon x \longrightarrow \frac{Px-1}{x+1-a} = P + \frac{(a-1)P-1}{x-a+1} \quad \text{for} \quad x \geq (a+1).$$

We can easily see that f is a decreasing function that assumes its maximum at $x = a+1$. Hence,

$$\alpha = \frac{Pq-1}{r-a} \leq f(q) = \frac{Pq-1}{q+1-a} \leq f(a+1) = \frac{P(a+1)-1}{2} < \frac{a+1}{2}P.$$

(3) Also, we claim that $q - p_{d-2} \geq 2$. Indeed if this is not the case, then $q - p_{d-2} = 1$. This is equivalent to ($q = 2$ and $p_{d-2} = 1$) or ($q = 3$ and $p_{d-2} = 2$). Hence, $N = 2r$ or $N = 6r$ is a K_a-number.

- Suppose that $N = 2r$ (i.e., $q = 2 < r$) is a K_a-number, then $0 < a < q = 2 < r$ which implies that $a = 1$. Hence by (F_1), we get $r - a = r - 1$ divides $2 - 1 = 1$. Therefore, $r - 1 = 1$ and so $r = 2$, a contradiction with $q < r$.
- Suppose now that $N = 6r$ (i.e., $q = 3 < r$) is a K_a-number. As $a \neq p_{d-2} = 2$ and $0 < a < q = 3 < r$ then $a = 1$. Hence by (F_1), we have $q - a = 2$ divides $N - a = 6r - 1$ which is odd, a contradiction.

Now, as $p_{d-2} + 2 \leq q$, we obtain

$$p_{d-2} + 2 - a \leq q - a = \frac{(aP-1)(P+\alpha)}{\Delta} \qquad (F_4).$$

(i) If $p_{d-2} + 2 - a \leq 0$, then we can write

$$1 \leq q - a = \frac{(aP-1)(P+\alpha)}{\Delta}$$

which is equivalent to

$$\Delta = \alpha\beta - P^2 \leq (aP-1)(P+\alpha).$$

Therefore by (2), we have

$$\Delta = \alpha\beta - P^2 \leq (aP-1)(P+\alpha) \leq (aP-1)\frac{a+3}{2}P.$$

Hence,

$$\alpha\beta \leq P^2 + (aP-1)\frac{a+3}{2}P < P^2\left(1 + \frac{a(a+3)}{2}\right) = \frac{(a+1)(a+2)}{2}P^2.$$

Finally, as $\Delta = \alpha\beta - P^2 > 0$, we conclude that

$$P^2 < \alpha\beta < \frac{(a+1)(a+2)}{2} P^2.$$

(ii) Suppose that $p_{d-2} + 2 - a > 0$. Then, by (F_4) and (2), we get

$$\Delta(p_{d-2} + 2 - a) \leq (aP - 1)(P + \alpha) \leq \frac{a+3}{2} P(aP - 1).$$

Thus,

$$\Delta = \alpha\beta - P^2 \leq \frac{P(a+3)(aP-1)}{2(p_{d-2} + 2 - a)},$$

and so

$$\alpha\beta < \frac{P^2(2p_{d-2} + 4 + a(a+1))}{2(p_{d-2} + 2 - a)}.$$

Finally, as $\Delta = \alpha\beta - P^2 > 0$, we obtain

$$P^2 < \alpha\beta < \frac{P^2(2p_{d-2} + 4 + a(a+1))}{2(p_{d-2} + 2 - a)}.$$

□

3. Factor Bounds Of A K_a-Number

In this section we derive upper bounds for q and r (so for N) in terms of a and P. As a consequence, for each fixed P and a there are only finitely many N that are K_a-numbers.

Theorem 3.1. *If $a < 0$, then*

$$\begin{cases} q & < & -2aP^2; \\ r & < & -2aP^3. \end{cases}$$

Proof. We consider two cases.

(1) If $\mid a \mid < q$, then $\gcd(a, q) = \gcd(a, r) = 1$. By Proposition 2.2, we have

$$q - a = \frac{(P+\alpha)(aP-1)}{\Delta}.$$

As $a < 0$, and by Proposition 2.2, $\Delta = \alpha\beta - P^2 \leq -1$ and $\alpha \leq P - 1$, then we can write

$$q = a + \frac{(P+\alpha)(1-aP)}{P^2 - \alpha\beta} \leq a + (P + P - 1)(1 - aP).$$

Thus,

$$q \leq -2aP^2 + (a+2)P + a - 1. \qquad (F_5)$$

As $a < 0$, we discuss two cases:

- If $a \leq -2$: It's obvious from (F_5) that $q \leq -2aP^2$.

- If $a = -1$:
First, we claim that $\alpha \leq P - 2$. Indeed, suppose that this is not the case, then by Proposition 2.2, we obtain $\alpha = P - 1$. So, by (F_1), we can write
$$Pq - 1 = (P - 1)(r + 1) = rP - r + P - 1.$$
Therefore, $Pq = P(r + 1) - r$, this implies that P divides r, a contradiction.
Now, with $a = -1$, Proposition 2.2 gives
$$q = -1 + \frac{(P + \alpha)(P + 1)}{P^2 - \alpha\beta} \leq -1 + (P + P - 2)(P + 1).$$
Thus,
$$q < 2P^2 = -2aP^2.$$

Then, we conclude that for all $a < 0$, we have $q < -2aP^2$.
Now, since $r - a \leq \alpha(r - a) = Pq - 1$, we obtain
$$r \leq Pq - 1 + a < Pq < -2aP^3.$$

(2) Suppose that $q \leq\mid a \mid$. Then clearly $q \leq\mid a \mid< -2aP^2$.
- If $\gcd(r, a) = 1$, then as $Pq - 1 = \alpha(r - a)$ with $1 \leq \alpha$, we obtain $r < r - a \leq \alpha(r - a) = Pq - 1 < P \mid a \mid\leq -2aP^3$.
- Now, if r divides a, we obtain
$$r \leq -a < -2aP^3.$$

Finally we conclude that, in all cases, we have $q < -2aP^2$ and $r < -2aP^3$. □

Theorem 3.2. *If $a > 0$, then*
$$\begin{cases} q & < & \dfrac{a(a+3)}{2}P^2; \\ r & < & \dfrac{a(a+3)}{2}P^3. \end{cases}$$

Proof. We have two cases to be considered.
(1) If $a < q$, then $\gcd(a, q) = \gcd(a, r) = 1$ and by Proposition 2.2, we have
$$q - a = \frac{(\alpha + P)(aP - 1)}{\Delta}.$$
Hence, by Proposition 2.4, $\Delta \geq 1$ and $\alpha < \dfrac{a+1}{2}P$.
Then, we obtain
$$q < a + \left(\frac{a+1}{2}P + P\right)(aP - 1) = \frac{a+3}{2}P(aP - 1) + a.$$

This gives

$$q < \frac{a(a+3)}{2}P^2 - \left(\frac{a+3}{2}P - a\right). \qquad (F_6)$$

But as $P \geq 1$, we consider the following three subcases:

(a) Suppose that $P \geq 3$. Then we have $\frac{a+3}{2}P > a+3 > a$, hence by (F_6), we obtain

$$q < \frac{a(a+3)}{2}P^2 - \left(\frac{a+3}{2}P - a\right) < \frac{a(a+3)}{2}P^2.$$

On the other hand, we have $Pq - 1 = \alpha(r-a)$ and $\alpha \geq 1$. Therefore, $r - a \leq Pq - 1$, so by (F_6), we obtain

$$r \leq Pq - 1 + a < \frac{a(a+3)}{2}P^3 - \left(\frac{a+3}{2}P^2 - aP - a + 1\right). \quad (F_7)$$

We claim that the quantity $\frac{a+3}{2}P^2 - aP - a + 1$ in (F_7) is positive. Indeed, define the function:

$$x \longrightarrow f(x) = (a+3)x^2 - 2ax - 2a + 2.$$

Let $\delta = 4\delta' = 4(3a^2 + 4a - 6) > 0$ and $\{P_1, P_2\}$ be respectively the discriminant and the solution set of the equation $f(x) = 0$. Then

$$P_1 = \frac{a - \sqrt{\delta'}}{a+3} \leq 0 < \frac{a + \sqrt{\delta'}}{a+3} = P_2.$$

As $\delta' = 3a^2 + 4a - 6 < 4a^2$, we have $P_2 = \frac{a+\sqrt{\delta'}}{a+3} < \frac{a+2a}{a+3} = \frac{3a}{a+3} < 3.$

By studying the sign of $f(P)$, we can easily see that $f(P) > 0$ for each $P \geq 3$. This implies that

$$\frac{a+3}{2}P^2 - aP - a + 1 = \frac{f(P)}{2} > 0 \text{ for each } P \geq 3.$$

Thus, by (F_7), we get

$$r < \frac{a(a+3)}{2}P^3.$$

(b) If $P = 2$, we consider two cases:

(i) If $\Delta = 1$, then $\alpha\beta - P^2 = 1$. As $P = 2$ and $\alpha < \beta$ then $\alpha = 1$ and $\beta = 5$ and by (F_1) we obtain

$$\begin{cases} 2q - 1 &= r - a; \\ 2r - 1 &= 5(q - a). \end{cases}$$

This implies that $q = 7a - 3$ and $r = 15a - 7$.

Let $g(a) = \dfrac{a(a+3)}{2}P^2 - q = 2a(a+3) - 7a + 3 = 2a^2 - a + 3$.
As the discriminant of $g(a)$ is $\delta = -23 < 0$, then $g(a) > 0$ for all $a > 0$. Hence, $q < \dfrac{a(a+3)}{2}P^2$.
Let $h(a) = \dfrac{a(a+3)}{2}P^3 - r = 4a^2 - 3a + 7$.
As the discriminant of $h(a)$ is $\delta = -103 < 0$, then $h(a) > 0$ for all $a > 0$. Hence, $r < \dfrac{a(a+3)}{2}P^3$.

(ii) Suppose that $\Delta \geq 2$.
By Proposition 2.2, we have $q - a = \dfrac{(\alpha + P)(aP-1)}{\Delta}$.
As $\Delta \geq 2$, $P = 2$ and by Proposition 2.4, $\alpha < \dfrac{a+1}{2}P$, we get

$$q = a + \frac{(\alpha+P)(aP-1)}{\Delta} < a + \frac{(\frac{a+1}{2}P + P)(aP-1)}{2}.$$

This implies that

$$q < \frac{2a^2+7a-3}{2}.$$

So, we obtain

$$q < \frac{2a^2+7a-3}{2} < 2a(a+3) = \frac{a(a+3)}{2}P^2.$$

Now, as $\alpha \geq 1$ and $Pq - 1 = \alpha(r-a)$, we can write

$$\begin{aligned} r &\leq Pq - 1 + a = 2q - 1 + a \\ &< 2(\frac{2a^2+7a-3}{2}) + a - 1 = 2a^2 + 8a - 4 \\ &< 4a(a+3) = \frac{a(a+3)}{2}P^3. \end{aligned}$$

(c) Now suppose that $P = 1$. We consider two cases:

(i) If $\Delta = 1$, then $\alpha\beta - P^2 = 1$. Therefore, as $P = 1$ and $0 < \alpha < \beta$, we get $\alpha = 1$ and $\beta = 2$. Hence by (F_1), we obtain

$$\begin{cases} q - 1 &= \quad r - a, \\ r - 1 &= \quad 2(q-a). \end{cases}$$

This implies that $q = 3a - 2$ and $r = 4a - 3$. But q and r are prime numbers, then $a \notin \{1, 2, 3, 4\}$. Hence, $a \geq 5$.
Let $\dfrac{g(a)}{2} = \dfrac{a(a+3)}{2}P^2 - q = \dfrac{a(a+3)}{2} - 3a + 2 = \dfrac{a^2-3a+4}{2}$.

As the discriminant of $g(a)$ is $\delta = -7 < 0$, then $g(a) > 0$ for all $a > 0$. Hence, $q < \frac{a(a+3)}{2}P^2$.

Let $\frac{h(a)}{2} = \frac{a(a+3)}{2}P^3 - r = \frac{a(a+3)}{2} - 4a + 3 = \frac{a^2 - 5a + 6}{2}$. As the discriminant of $h(a)$ is $\delta = 1$, then $h(a) = 0$ if and only if $a = 2$ or $a = 3$. But $a \geq 5$, then $h(a) > 0$. Hence, $r < \frac{a(a+3)}{2}P^3$.

(ii) Suppose that $\Delta \geq 2$.

By Proposition 2.2, we have $q - a = \frac{(\alpha + P)(aP - 1)}{\Delta}$.

As $\Delta \geq 2$, $P = 1$ and by Proposition 2.4, $\alpha < \frac{a+1}{2}P$, we get

$$q = a + \frac{(\alpha + P)(aP - 1)}{\Delta} < a + \frac{\left(\frac{a+1}{2}P + P\right)(aP - 1)}{2}.$$

This implies that

$$q < \frac{a^2 + 6a - 3}{4}.$$

Hence,

$$q < \frac{a^2 + 6a - 3}{4} < \frac{a(a+3)}{2} = \frac{a(a+3)}{2}P^2.$$

As $Pq - 1 = \alpha(r - a), \alpha \geq 1$ and $P = 1$, then we can write $q - 1 \geq r - a$. Therefore, $r \leq q - 1 + a < \frac{a^2 + 6a - 3}{4} + a - 1 = \frac{a^2 + 10a - 7}{4}$.

Finally, we obtain $r < \frac{a(a+3)}{2} = \frac{a(a+3)}{2}P^3$.

(2) Suppose that $q < a$. At first, as $P \geq 1$, it's clear to see that $q < a < \frac{a(a+3)}{2}P^2$.

(a) Suppose that $a < r$. Then $\gcd(r, a) = 1$, and as $Pq - 1 = \alpha(r - a)$ with $\alpha \geq 1$, we get

$$r = \frac{Pq - 1}{\alpha} + a \leq Pq - 1 + a \leq P(a - 1) + a - 1.$$

Hence, $r \leq (a - 1)(P + 1) < \frac{a(a+3)}{2}P^3$.

(b) Now, suppose that $r < a$. Then $r < a < \frac{a(a+3)}{2}P^3$.

Finally we conclude that, in all cases, we have $q < \dfrac{a(a+3)}{2}P^2$ and $r < \dfrac{a(a+3)}{2}P^3$. □

Remark 3.3. *If $a = 1$ which is the case of Carmichael numbers, we can give an improvement to the bounds in Theorem 3.2 as given in* [5] *as follows.*

We have $Pq - 1 = \alpha(r-1)$. We claim that $\alpha \geq 2$. Indeed, if it is not the case, then, as $\alpha > 0$, $Pq - 1 = r - 1$. Thus $Pq = r$, which contradicts the primality of r. Hence, $r \leq \dfrac{Pq-1}{2} + 1$, but as $q < \dfrac{a(a+3)}{2}P^2 = 2P^2$, we get

$$r \leq \frac{P(2P^2-1)-1}{2} + 1 < P^3.$$

From Theorem 3.1 and Theorem 3.2, we immediately obtain the following theorem.

Theorem 3.4. *Let $a \in \mathbb{Z} - \{0\}$ and $p_1 < p_2 < \cdots < p_{d-2}$ be a given set of $d-2$ primes, $d \geq 3$. Then there are only finitely many K_a-numbers $N = \prod_{i=1}^{d} p_i$, where p_{d-1} and p_d are primes such that $p_{d-2} < p_{d-1} < p_d$.*

Proof. By Theorem 3.1 and Theorem 3.2, respectively we have:
- If $a < 0$, then $N = Pqr < 4a^2P^6$.
- If $a > 0$, then $N = Pqr < \dfrac{a^2(a+3)^2}{4}P^6$. □

4. W_a-Numbers

In this section we prove that for each fixed a there are only finitely many W_a-numbers with three prime factors, by handling separately the cases $p_1 < a$ and $p_1 > a$.

Proposition 4.1 (Characterization of W_a-numbers)**.** *Let a be a nonzero positive integer. Let $N = \prod_{i=1}^{k} p_i$ be a composite squarefree integer and $d_i = \gcd(a, p_i)$ for each $i \in \{1, \ldots, k\}$.*

N is a W_a-number if and only if $\dfrac{p_i^2 - a^2}{d_i^2}$ divides $2\left(\dfrac{N}{p_i} - 1\right)$ for each $i \in \{1, \ldots, k\}$.

Proof. Suppose that N is a W_a-number. We note that $p_i - a$ divides $N - a = N - p_i + p_i - a$ is equivalent to

$$p_i - a \ \text{ divides } \ N - p_i = p_i\left(\frac{N}{p_i} - 1\right). \qquad (F_8)$$

Since $d_i = \gcd(a, p_i) \in \{1, p_i\}$,

- If $d_i = 1$ and as $\gcd(p_i - a, p_i) = 1$, then (F_8) is equivalent to
$$p_i - a = \frac{p_i - a}{d_i} \text{ divides } \frac{N}{p_i} - 1.$$
- If $d_i = p_i$, then (F_8) is equivalent to
$$\frac{p_i - a}{p_i} = \frac{p_i - a}{d_i} \text{ divides } \frac{p_i}{d_i}\left(\frac{N}{p_i} - 1\right) = \frac{N}{p_i} - 1.$$

Thus, $p_i - a$ divides $N - a$ is equivalent to $\frac{p_i - a}{d_i}$ divides $\frac{N}{p_i} - 1$.

In a similar way, we deduce that $p_i + a$ divides $N + a$ is equivalent to $\frac{p_i + a}{d_i}$ divides $\frac{N}{p_i} - 1$.

Hence, N is a W_a-number is equivalent to both
$$\frac{p_i - a}{d_i} \text{ and } \frac{p_i + a}{d_i} \text{ divide } \frac{N}{p_i} - 1. \quad (F_9)$$

This implies that $\frac{(p_i - a)(p_i + a)}{D_i d_i^2}$ divides $\frac{N}{p_i} - 1$ with
$$D_i = \gcd\left(\frac{p_i - a}{d_i}, \frac{p_i + a}{d_i}\right).$$

On the other hand, as D_i divides both $\frac{p_i - a}{d_i}$ and $\frac{p_i + a}{d_i}$, then D_i divides $2\frac{p_i}{d_i} = \frac{p_i - a}{d_i} + \frac{p_i + a}{d_i}$ and D_i divides $2\frac{a}{d_i} = \frac{p_i + a}{d_i} - \frac{p_i - a}{d_i}$. Therefore, D_i divides $2\gcd(\frac{a}{d_i}, \frac{p_i}{d_i}) = 2$.

So we conclude that, if N is a W_a-number then $\frac{p_i^2 - a^2}{d_i^2}$ divides $2\left(\frac{N}{p_i} - 1\right)$.

Conversely, suppose that $\frac{p_i^2 - a^2}{d_i^2}$ divides $2\left(\frac{N}{p_i} - 1\right)$.

- If $\gcd\left(2, \frac{p_i^2 - a^2}{d_i^2}\right) = 1$, then $\frac{p_i^2 - a^2}{d_i^2}$ divides $\frac{N}{p_i} - 1$.
- Now, suppose that $\gcd\left(2, \frac{p_i^2 - a^2}{d_i^2}\right) = 2$, then 2 divides $\frac{p_i - a}{d_i}$ or $\frac{p_i + a}{d_i}$. But as
$$\frac{p_i - a}{d_i} = \left(\frac{p_i + a}{d_i}\right) - 2\frac{a}{d_i},$$
then 2 divides $\frac{p_i - a}{d_i}$ if and only if 2 divides $\frac{p_i + a}{d_i}$.

So,

$$\frac{p_i^2-a^2}{2d_i^2}=\left(\frac{p_i-a}{2d_i}\right)\left(\frac{p_i+a}{d_i}\right)=\left(\frac{p_i-a}{d_i}\right)\left(\frac{p_i+a}{2d_i}\right)$$

divides $\dfrac{N}{p_i}-1$.

This implies that both $\dfrac{p_i-a}{d_i}$ and $\dfrac{p_i+a}{d_i}$ divides $\dfrac{N}{p_i}-1$.

Finally, by (F_9), we conclude that N is a W_a-number.

□

Let $N=p_1p_2p_3$ be a W_a-number such that $a<p_1<p_2<p_3$. As $\gcd(a,p_i)=1$ for each $i\in\{1,2,3\}$, then by Proposition 4.1 there exist positive integers α, β and γ such that

$$\begin{cases} 2p_2p_3-2 &= \alpha(p_1^2-a^2), & (E_1)\\ 2p_1p_3-2 &= \beta(p_2^2-a^2), & (E_2)\\ 2p_1p_2-2 &= \gamma(p_3^2-a^2). & (E_3)\end{cases}$$

Lemma 4.2. (1) $0<\gamma<\beta<\alpha$.

(2) $p_3<\dfrac{\beta(\gamma+1)}{\gamma(\beta+1)}p_2$.

(3) $\gamma p_3^3<\beta p_2^3<\alpha p_1^3$.

(4) $8<\alpha\beta\gamma$.

Proof. (1) As $a<p_1<p_2<p_3$ we have

$$0<p_1^2-a^2<p_2^2-a^2<p_3^2-a^2,$$

and

$$0<2(p_1p_2-1)<2(p_1p_3-1)<2(p_2p_3-1).$$

Then

$$0<\gamma=\frac{2(p_1p_2-1)}{p_3^2-a^2}<\beta=\frac{2(p_1p_3-1)}{p_2^2-a^2}<\alpha=\frac{2(p_2p_3-1)}{p_1^2-a^2}.$$

(2) The equation $\beta(E_3)-\gamma(E_2)$ gives

$$2p_1(\beta p_2-\gamma p_3)=\beta\gamma(p_3^2-p_2^2)+2(\beta-\gamma)$$

Thus,

$$2p_1((\beta-\gamma)p_2-\gamma(p_3-p_2))=\beta\gamma(p_3^2-p_2^2)+2(\beta-\gamma),$$

which implies that

$$2(\beta-\gamma)p_1p_2=2\gamma p_1(p_3-p_2))+\beta\gamma(p_3^2-p_2^2)+2(\beta-\gamma).$$

So

$$p_1p_2=\frac{(2\gamma p_1+\beta\gamma(p_3+p_2))(p_3-p_2)}{2(\beta-\gamma)}+1.$$

As $p_3 + p_2 > 2p_1$, then

$$p_1p_2 > \frac{(2\gamma + 2\beta\gamma)p_1(p_3 - p_2)}{2(\beta - \gamma)} + 1 > \frac{\gamma(\beta + 1)(p_3 - p_2)}{\beta - \gamma}p_1.$$

Thus,

$$p_2 > \frac{\gamma(\beta + 1)}{\beta - \gamma}(p_3 - p_2),$$

and hence,

$$\frac{\gamma(\beta + 1)}{\beta - \gamma}p_3 < p_2(1 + \frac{\gamma(\beta + 1)}{\beta - \gamma}),$$

and so

$$p_3 < (1 + \frac{\beta - \gamma}{\gamma(\beta + 1)})p_2 = \frac{\beta(\gamma + 1)}{\gamma(\beta + 1)}p_2.$$

(3) The equations $\beta(E_3) - \gamma(E_2)$ and $\alpha(E_2) - \beta(E_1)$ give respectively

$$2p_1(\beta p_2 - \gamma p_3) = \beta\gamma(p_3^2 - p_2^2) + 2(\beta - \gamma)$$

and

$$2p_3(\alpha p_1 - \beta p_2) = \alpha\beta(p_2^2 - p_1^2) + 2(\alpha - \beta).$$

As $0 < \gamma < \beta < \alpha$ and $p_1 < p_2 < p_3$, we obtain

$$\gamma p_3 < \beta p_2 < \alpha p_1. \qquad (F_{10})$$

On the other hand, the equations (E_1), (E_2), and (E_3) are equivalent to

$$\begin{cases} 2p_2p_3 = \alpha(p_1^2 - a^2) + 2, & (E_4) \\ 2p_1p_3 = \beta(p_2^2 - a^2) + 2, & (E_5) \\ 2p_1p_2 = \gamma(p_3^2 - a^2) + 2. & (E_6) \end{cases}$$

The division of (E_4) by (E_5) gives $\dfrac{p_2}{p_1} = \dfrac{\alpha(p_1^2 - a^2) + 2}{\beta(p_2^2 - a^2) + 2}$, and so

$$p_1(\alpha(p_1^2 - a^2) + 2) = p_2(\beta(p_2^2 - a^2) + 2).$$

It follows that

$$\alpha p_1^3 - \beta p_2^3 = a^2(\alpha p_1 - \beta p_2) + 2(p_2 - p_1) > 0.$$

As $p_1 < p_2$ and by (F_{10}) we have $\beta p_2 < \alpha p_1$, then $\beta p_2^3 < \alpha p_1^3$.

With the same idea, the division of (E_5) by (E_6) gives

$$\beta p_2^3 - \gamma p_3^3 = a^2(\beta p_2 - \gamma p_3) + 2(p_3 - p_2).$$

As $p_2 < p_3$ and by (F_{10}) we have $\gamma p_3 < \beta p_2$, then $\gamma p_3^3 < \beta p_2^3$.

Finally we conclude that

$$\gamma p_3^3 < \beta p_2^3 < \alpha p_1^3.$$

(4) As $\gamma p_3^3 < \beta p_2^3 < \alpha p_1^3$, we obtain

$$p_3\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{3}} < p_1 \text{ and } p_3\left(\frac{\gamma}{\beta}\right)^{\frac{1}{3}} < p_2. \qquad (F_{11})$$

As by (E_3), we have $2(p_1p_2-1)=\gamma(p_3^2-a^2)$, then (F_{11}) gives

$$2\left(\frac{\gamma^{\frac{2}{3}}}{(\alpha\beta)^{\frac{1}{3}}}p_3^2-1\right) < 2(p_1p_2-1)=\gamma(p_3^2-a^2).$$

Therefore,

$$\frac{\gamma^{\frac{2}{3}}(2-(\alpha\beta\gamma)^{\frac{1}{3}})}{(\alpha\beta)^{\frac{1}{3}}}p_3^2 < 2-\gamma a^2. \qquad (F_{12})$$

Two cases are to be considered:

- If $a \geq 2$, then $2-\gamma a^2 < 0$ and by (F_{12}) we obtain $2 < (\alpha\beta\gamma)^{\frac{1}{3}}$. Hence, $8 < \alpha\beta\gamma$.
- Suppose that $a=1$ and $\alpha\beta\gamma \leq 8$. Then

  $$(\alpha,\beta,\gamma) \in \{(3,2,1),(4,2,1)\}.$$

  Then by (E_2), we get $2(p_1p_3-1)=2(p_2^2-1)$. Therefore, $p_1p_3=p_2^2$ which implies that $p_1=p_2=p_3$, a contradiction.

So we conclude that

$$8 < \alpha\beta\gamma.$$

□

Theorem 4.3. *Let a be a nonzero positive integer. There exist only finitely many W_a-numbers with three prime factors.*

Proof. Let a be a fixed positive integer and $N=p_1p_2p_3$ be a W_a-number such that $p_1<p_2<p_3$.

Two cases are to be considered:

– If $p_1 < a$, then there is a finite number of possibilities for p_1. For each possibility for p_1, and by Theorem 3.4 there are only finitely many K_{-a}-numbers and K_a-numbers $N=p_1p_2p_3$. Hence, there are only finitely many W_a-numbers $N=p_1p_2p_3$ with $p_1<a$.

– Now, suppose that $a < p_1$. By Lemma 4.3, we have $8 < \alpha\beta\gamma$, this leads us to discuss the two following cases.

Case 1: If $(\gamma,\beta)=(1,2)$.

(a) Suppose that $\alpha=5$.
The relation $(E_1)+(E_2)+(E_3)$ gives

$$5p_1^2+2p_2^2+p_3^2-2p_1p_2-2p_1p_3-2p_2p_3=8a^2-6.$$

Thus,

$$5\left(p_1 - \frac{p_2+p_3}{5}\right)^2 + \frac{9}{5}\left(p_2 - \frac{2}{3}p_3\right)^2 = 8a^2 - 6.$$

It follows that

$$\frac{9}{5}\left(p_2 - \frac{2}{3}p_3\right)^2 < 8a^2 - 6. \qquad (F_{13})$$

But, as $p_3 < \dfrac{\beta(\gamma+1)}{\gamma(\beta+1)}p_2 = \dfrac{4}{3}p_2$ and by (F_{13}), we obtain

$$\frac{1}{5}\left(\frac{p_3}{4}\right)^2 < \frac{9}{5}\left(p_2 - \frac{2}{3}p_3\right)^2 < 8a^2.$$

Therefore,

$$p_3 < 8\sqrt{10}a.$$

(b) Now, suppose that $\alpha \geq 6$, then by equation (E_1) we obtain

$$6(p_1^2 - a^2) \leq 2(p_2p_3 - 1). \qquad (F_{14})$$

Now, the relation $(F_{14}) + (E_2) + (E_3)$ gives

$$6(p_1^2-a^2)+2(p_2^2-a^2)+(p_3^2-a^2) \leq 2(p_2p_3-1)+2(p_1p_3-1)+2(p_1p_2-1).$$

This implies that

$$6\left(p_1 - \frac{p_2+p_3}{6}\right)^2 + \frac{11}{6}\left(p_2 - \frac{7}{11}p_3\right)^2 + \frac{p_3^2}{11} \leq 9a^2 - 6. \qquad (F_{15})$$

On the other hand as $p_3 < \dfrac{\beta(\gamma+1)}{\gamma(\beta+1)}p_2 = \dfrac{4}{3}p_2$, we have

$$\frac{11}{6}\left(p_2 - \frac{7}{11}p_3\right)^2 > \frac{25}{1056}p_3^2.$$

Hence,

$$\frac{11}{6}\left(p_2 - \frac{7}{11}p_3\right)^2 + \frac{p_3^2}{11} > \frac{25}{1056}p_3^2 + \frac{p_3^2}{11} = \frac{97}{1056}p_3^2.$$

Then by (F_{15}), we obtain

$$\frac{97}{1056}p_3^2 < 9a^2,$$

and so

$$p_3 < 10a.$$

Case 2: Now, suppose that $(\gamma, \beta) \neq (1, 2)$ which is equivalent to $(\alpha \geq 4, \beta \geq 3$ and $\gamma \geq 1)$.
The equations (E_1), (E_2), and (E_3) give respectively

$$4(p_1^2 - a^2) \leq 2(p_2p_3 - 1), \qquad (E_7)$$

$$3(p_2^2 - a^2) \leq 2(p_1p_3 - 1), \qquad (E_8)$$

$$p_3^2 - a^2 \leq 2(p_1p_2 - 1). \qquad (E_9)$$

Then the relation $(E_7) + (E_8) + (E_9)$ gives

$$4(p_1^2 - a^2) + 3(p_2^2 - a^2) + (p_3^2 - a^2) \leq 2(p_2p_3 - 1) + 2(p_1p_3 - 1) + 2(p_1p_2 - 1).$$

This implies that

$$4\left(p_1 - \frac{p_2 + p_3}{4}\right)^2 + \frac{11}{4}\left(p_2 - \frac{5}{11}p_3\right)^2 + \frac{2p_3^2}{11} \leq 8a^2 - 6. \qquad (F_{16})$$

On the other hand as $p_3 < \frac{\beta(\gamma+1)}{\gamma(\beta+1)}p_2 \leq 2p_2$, we have

$$\frac{11}{4}\left(p_2 - \frac{5}{11}p_3\right)^2 > \frac{1}{176}p_3^2.$$

Therefore,

$$\frac{11}{4}\left(p_2 - \frac{5}{11}p_3\right)^2 + \frac{2p_3^2}{11} > \frac{1}{176}p_3^2 + \frac{2p_3^2}{11} = \frac{363}{1936}p_3^2.$$

Then by (F_{16}), we get

$$\frac{363}{1936}p_3^2 < 8a^2,$$

and so

$$p_3 < 7a.$$

Thus, in all cases, p_3 is bounded. Since $p_1 < p_2 < p_3$, the number of possibilities for $N = p_1p_2p_3$ such that $a < p_1$ is finite.

Finally, we conclude that for each fixed a there are only finitely many W_a-numbers with three prime factors. $\square$

5. Acknowledgement

We thank the referee for his/her report improving both presentation and the mathematical content of the paper.

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MSC2000: 11Y16, 11Y11, 11A51.

Key words and phrases: Carmichael number, Korselt number, Williams number, Prime number, squarefree composite number.

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