CONSTRUCTION OF AN ORDINARY DIRICHLET SERIES WITH CONVERGENCE BEYOND THE BOHR STRIP

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Abstract. An ordinary Dirichlet series has three abscissae of interest, describing the maximal regions where the Dirichlet series converges, converges uniformly, and converges absolutely. The paper of Hille and Bohnenblust in 1931, regarding the region on which a Dirichlet series can converge uniformly but not absolutely, has prompted much investigation into this region, the "Bohr strip." However, a related natural question has apparently gone unanswered: For a Dirichlet series with non-trivial Bohr strip, how far beyond the Bohr strip might the series converge? We investigate this question by explicit construction, creating Dirichlet series which converge beyond their Bohr strip.

1. Introduction

An ordinary Dirichlet series is a function of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with $s = \sigma + it \in \mathbb{C}$. The region on which a Dirichlet series might be expected to converge is a right half plane, we denote these by

$$\Omega_\sigma = \{s \in \mathbb{C} : \Re s > \sigma\}$$

(where $\Re$ denotes the real part) and its closure will be written $\overline{\Omega}_\sigma$. To a Dirichlet series we can associate several abscissae:

$$\sigma_a = \inf\{\sigma : \sum a_n n^{-s} \text{ converges absolutely for } s \in \Omega_\sigma\}$$

$$\sigma_b = \inf\{\sigma : \sum a_n n^{-s} \text{ converges to a bounded function on } \Omega_\sigma\}$$

$$\sigma_c = \inf\{\sigma : \sum a_n n^{-s} \text{ converges for all } s \in \Omega_\sigma\}\ .$$

From the definitions, it is evident that $\sigma_c \leq \sigma_b \leq \sigma_a$. Harald Bohr proved that $\sigma_a - \sigma_b \leq 1/2$ in ([4], Satz X), although as noted in [6] this now follows relatively easily from a Parseval-type inequality (see for example [11], top of p. 156). In 1931, Hille and Bohnenblust [3] showed that this is sharp; there exist Dirichlet series for which $\sigma_a - \sigma_b = 1/2$, and an explicit construction is provided in [3] (where the crucial construction of multi-variable polynomials is given by [3], Theorem IV, and this construction is applied to the Dirichlet series in Sections 5 and 6).

Let $\Omega(n)$ be the number of prime factors of $n \in \mathbb{N}$, counted with multiplicity (so $\Omega(8) = 3$). In [3], Theorems V and VI also show that if $\sum a_n n^{-s}$ contains only terms of homogeneity at most M, i.e.

$$\Omega(n) > M \implies a_n = 0$$

then we have $\sigma_a - \sigma_b \leq \frac{1}{2} - \frac{1}{2M}$, and this is also sharp, which is shown by construction.

Since the publication of [3], there has been much investigation into the gap $\sigma_a - \sigma_b$, and the associated "Bohr strip" $\{s \in \mathbb{C} : \sigma_b < \Re s < \sigma_a\}$ and related issues, we recall some of them here. We would like to mention a survey article in this area by Defant and Schwarting [6], as well as the discussion in [10].

A key inequality in studying the Bohr strip is the following result for M-homogenous polynomials: For each M, there is a constant D_M such that, for an M-homogenous polynomial $\sum_{|\alpha|=M} a_\alpha z^\alpha$ on $\mathbb{C}^n$ we have

$$\left(\sum_{|\alpha|=M} |a_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq D_M \sup_{z \in \overline{\mathbb{D}}^n} \left| \sum_{|\alpha|=M} a_\alpha z^\alpha \right| \tag{1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\overline{\mathbb{D}}$ is the closed unit disc. This is proved in this form in [7], although the original proof is in [3]. Note that the result in [3] is not stated in this form, (1) must be deduced from the multi-linear version ([3] Theorem I) and the discussion at the beginning of [3], Section 3. The result in (1) is a generalization of the Littlewood 4/3 inequality [12], which proves the above result in the case $M = 2$. From the original proof in [3], one can derive a bound on the best possible D_M, but this bound has been substantially improved, see the discussions in [7] and [8] which also contain the most recent improvements to our knowledge.

Another development related to the question of the gap $\sigma_a - \sigma_b$ is the theory of p-Sidon sets (see the discussions in [14, 7]), the inequality in (1) shows that the set of monomials $\{z^\alpha : |\alpha| = M\}$ is a $\frac{2M}{M+1}$-Sidon set, for example.

To produce Dirichlet series with a large gap $\sigma_a - \sigma_b$, in addition to the explicit construction of [3], random methods have been employed, such as in [9] where the existence of ordinary Dirichlet series with $\sigma_a - \sigma_b = 1/2$, (and $\sigma_a - \sigma_b = \frac{1}{2} - \frac{1}{2M}$ for homogeneity M), is shown. Random methods are also employed in Sections 4 and 5 of [14], to construct Dirichlet polynomials with small $\|\cdot\|_\infty$ norm and thus obtain bounds on the 1-Sidon constant of the set of "frequencies" $\{\log 1, \ldots, \log N\}$ (interpreted as functions on the Bohr compactification of $\mathbb{R}$).

We mention these developments to note that, for all of this progress, a rather natural question remains: For a Dirichlet series with the "gap" $\sigma_a - \sigma_b$ being large, what can be said about the "gap" $\sigma_b - \sigma_c$ for this same series? If a Dirichlet series has a large Bohr strip, to what extent can this series converge beyond its Bohr strip? This is the question we explore here.

We first present a general construction of an ordinary Dirichlet series of homogeneity M. Our technique is based on the method of Walsh matrices used in [13], although we depart from [13] by using non-square matrices. We then prove four bounds on the abscissae of this series, of the following forms:

- $\sigma_b \leq B$ [Proposition 1, Section 5]
- $\sigma_a \geq A$ [Proposition 2, Section 6]
- $\sigma_b \geq 0$ [Proposition 8, Section 7]
- $\sigma_c \leq C$ [Proposition 10, Section 9] .

The construction only yields a non-trivial result (i.e. $\sigma_a - \sigma_b > 0$, $\sigma_b - \sigma_c > 0$) for the cases $M = 2, 3$. We present the construction for general M nevertheless, because the exposition would not be much clearer for $M = 3$ rather than general M, and because we hope that better bounds might be proved for the general construction which would then yield results beyond $M = 3$. For the cases $M = 2, 3$, we obtain the following.

$M = 2$. We construct a Dirichlet series of homogeneity $M = 2$ satisfying

$$\sigma_a - \sigma_b = 1/4 \ , \quad \sigma_b - \sigma_c \geq 1/4 \ .$$

Construction of such a Dirichlet series (or even proof of its existence) is, to our knowledge, a new result. Note that $1/4$ is the optimal value of $\sigma_a - \sigma_b$, given that $M = 2$.

$M = 3$. Here, for any value $\rho_1 \in (0, 1)$, we construct a Dirichlet series of homogeneity $M = 3$ which satisfies $\sigma_a - \sigma_c \geq 1/3$, and we furthermore have some specific control over $\sigma_b \in (\sigma_c, \sigma_a)$:

$$\sigma_a - \sigma_b \geq \frac{1 + \rho_1}{6} \ , \quad \sigma_b - \sigma_c \geq \frac{1 - \rho_1}{9} \ .$$

For $\rho_1 > 1/2$, we see that the value of $\sigma_a - \sigma_b$ is larger than $1/4$, so this construction does represent a result that cannot be achieved with only terms of homogeneity at most two. If we pick, for example, $\rho_1 = 3/4$, then we have

$$\sigma_a - \sigma_b \geq 7/24 \ , \quad \sigma_b - \sigma_c \geq 1/36 \ .$$

Note that for $M = 3$, unfortunately the current construction does not produce values for $\sigma_a - \sigma_b$, $\sigma_b - \sigma_c$ that couldn't be replicated by a Dirichlet series with existing constructions. For instance, using the standard Hille-Bohnenblust construction for $M = 3$ and adding a properly shifted Dirichlet series with a given value of $\sigma_a - \sigma_c$ (such as the alternating zeta function) will produce a series that has these properties. This can be done using only terms of homogeneity three as well; simply use a version of the alternating zeta function which contains only terms of homogeneity three (and is "alternating" on these terms), we leave details to the interested reader. However, such a series, being more simply constructed, does not afford control on the individual coefficients. To our knowledge, ours is the first construction which exhibits a Dirichlet series having terms of homogeneity exactly three for which it is proved that $\sigma_a - \sigma_b > 1/4$, $\sigma_b - \sigma_c > 0$ and for which we have substantial knowledge regarding the individual coefficients.

Our hope is that the method shown here, since it gives specific control over each abscissa, without any "tricks" of adding another (unrelated) Dirichlet series, could be extended, specifically by improving the estimate in Section 8.

In Sections 3 and 4 we present the general construction. In Sections 5 and 6, we prove the "easy" bounds: an upper bound on σ_b and a lower bound on σ_a. These first two bounds yield the classic Hille-Bohnenblust-type Dirichlet series, for each homogeneity M. In Sections 7, 8 and 9 we prove the "hard" bounds, showing that our Dirichlet series is unbounded on any Ω_σ with $\sigma < 0$, and then showing that our Dirichlet series converges (conditionally) at a point $s = -\epsilon$ on the negative real axis. Once all four bounds are proved, in Section 10 we derive the results for $M = 2, 3$.

In Section 8 we isolate one of the key estimates, a basic size estimate on all partial sums of a certain set of complex numbers of modulus one. It seems some improvement should be possible, given that the arguments are spread over the unit circle.

2. Notation and Preliminaries

The list that follows is not meant necessarily to define these quantities, but rather to be used as a reference.

- $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$
- p_k The kth prime
- M An integer specifying the homogeneity of the construction

- J_M An integer depending only on M, defined in eq. (19), based on a general bound on M-homogenous polynomials
- L A positive integer index
- L' A positive integer, depending only on σ and M, defined in eq. (18)
- ρ_i, ρ Positive numbers, parameters of the construction, $\rho = \sum \rho_i$
- r_j An integer, $r_j = r_j(L)$ is the "length" of the jth "dimension" of Q^L
- $k_i^{(j)}(L)$ An integer, used as an index for a prime number
- $\Pi_L^{\times}$ A set of integers, each integer being a product of M prime numbers
- ω_r $r \in \mathbb{N}$, this equals $e^{2\pi i/r}$
- Q^L A multivariable polynomial, defined below
- P_L A Dirichlet polynomial created by substituting into Q^L
- X A real-valued parameter used to adjust the abscissae, fixed in equation (22)
- β_L A complex number of modulus one, determined in Section 9
- $A_N(\epsilon)$ A partial sum of the constructed Dirichlet series f (defined in (15), (16)), at the point $s = -\epsilon$

For a polynomial q in complex variables $z_1, z_2, \ldots$ we define

$$\|q\|_\infty = \max\{|q| : |z_i| \le 1\} \quad \text{[Infinity Norm on the Polydisc]} .$$

We let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{C}^n$, $\|x\|^2 = \sum |x_i|^2$.

Let p_k be the kth prime number. We will need the following result about the distribution of prime numbers: there exist $0 < c \le C$ such that

$$ck \log k \le p_k \le Ck \log k \ .$$

(see [1], Chapter 4 for this particular result, Chapter 13 for the Prime Number Theorem).

We denote the floor and ceiling functions for $x \in \mathbb{R}$ by $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \le x\}$, $\lceil x \rceil = \min\{n \in \mathbb{N} : n \ge x\}$. Next, we define some key parameters of this construction:

$$\begin{aligned} &L \in \mathbb{N} \\ &\rho_1 \le \cdots \le \rho_{M-1} \in [0,1],\ \rho_M = 1 \\ &\rho = \rho_1 + \cdots + \rho_M \\ &r_1 = \lfloor 2^{\rho_1 L} \rfloor, \ldots, r_M = \lfloor 2^{\rho_M L} \rfloor \end{aligned}$$

where $\rho_1, \ldots, \rho_M$ are fixed and L is an index which will range over $\mathbb{N}$. Notice that the r_j depend on L, so to be proper we might write $r_j^{(L)}$; we will not do so since the value of L will be clear from context. The ρ_i are parameters of the construction, controlling the length of different "dimensions" of the polynomial Q^L (and P_L), explained below. The r_j are the actual integer values for the lengths of each dimension.

Now, we define a family of disjoint sets of primes: For $L \in \mathbb{N}$, and $j = 1, \ldots, M$, let the sets $K_L^{(j)}$ be defined by

$$K_L^{(j)} = \left\{(M + j - 1)2^L + i \ : \ i = 0, \ldots, r_j - 1\right\}$$

and then the family of sets of primes is defined by

$$\Pi_L^{(j)} = \{p_k : k \in K_L^{(j)}\} \ .$$

For convenience, when the value of L is clear from context, we denote the ith element of $K_L^{(j)}$ by

$$k_i^{(j)} = (M + j - 1)2^L + i \ .$$

Note that all of the $\Pi_L^{(j)}$ are pairwise disjoint.

Define

$$\begin{aligned}\Pi_L^{\times} &= \Pi_L^{(1)} \cdot \Pi_L^{(2)} \cdots \Pi_L^{(M)} \\ &= \left\{ n = p_{k_{i_1}^{(1)}} p_{k_{i_2}^{(2)}} \cdots p_{k_{i_M}^{(M)}} \ , \quad i_j \in \{0, \ldots, r_j - 1\} \right\}, \ L \in \mathbb{N}.\end{aligned}$$

The terms in the Dirichlet polynomial P_L will involve only those n which are a product of a single prime from each $\Pi_L^{(j)}$, i.e. $n \in \Pi_L^{\times}$.

Note that, if $L_1 < L_2$, then the largest element of $\Pi_{L_1}^{(M)}$ is smaller than the smallest element of $\Pi_{L_2}^{(1)}$, because the largest element of $K_{L_1}^{(M)}$ is smaller than the smallest element of $K_{L_2}^{(1)}$ by construction: The largest element of $K_{L_1}^{(M)}$ is $k_{r_M}^{(M)}$, and

$$\begin{aligned}k_{r_M}^{(M)} &= (M + (M) - 1) * 2^{L_1} + r_M - 1 \\ &\leq (2M - 1) * 2^{L_1} + 2^{L_1} - 1 \\ &= 2M * 2^{L_1} - 1 \\ &< M * 2^{(L_1+1)}\end{aligned}$$

and $M * 2^{(L_1+1)}$ is the smallest element of $K_{L_1+1}^{(1)}$. This implies equation (8) below; in particular it means that the $\Pi_L^{\times}$ are disjoint for different L.

We collect here certain equations and inequalities that will be used repeatedly during the proof, or are purposeful features of the construction:

$$r_1 = \lfloor 2^{\rho_1 L} \rfloor, \ldots, r_M = \lfloor 2^{\rho_M L} \rfloor \tag{2}$$
$$2^{\rho_j L}/2 \le r_j \le 2^{\rho_j L} \tag{3}$$
$$ck \log k \le p_k \le Ck \log k, \; 0 < c \le C \tag{4}$$
$$M 2^L \le k_i^{(j)}(L) < M 2^{L+1} \tag{5}$$
$$n \in \Pi_L^\times \implies c_M 2^{LM} \le n \le C_M 2^{LM} L^M \tag{6}$$
$$\begin{aligned} \left|\Pi_L^\times\right| &= |\Pi_L^{(1)}| \cdots |\Pi_L^{(M)}| = |K_L^{(1)}| \cdots |K_L^{(M)}| \\ &= r_1 \cdots r_M \ge 2^{-M} 2^{\rho L} \quad \text{[using (3)]} \end{aligned} \tag{7}$$

The largest element of $\Pi_{L_1}^\times$ is smaller than the least element of $\Pi_{L_2}^\times$,
if $L_1 < L_2$ (8)

We will make use of summation by parts, in the following form: Suppose $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$ are given. Define $B_N = \sum_{j=1}^N b_j$. Then

$$\sum_{j=1}^{p} a_j b_j = \sum_{j=1}^{p-1} (a_j - a_{j+1}) B_j + a_p B_p \,.$$

If the a_j are non-decreasing and positive, then we have

$$\left| \sum_{j=1}^{p} a_j b_j \right| \le \max |B_j| * 2|a_p| \,. \tag{9}$$

Furthermore, we will let $c_1, c_2, \ldots$ denote unspecified positive real numbers that are either absolute or depend only on M.

3. Construction of the Polynomial Q

We will construct a multivariable polynomial Q with certain properties; Q will be used to construct Dirichlet polynomials. The construction here differs from standard constructions of this type, because the matrices we use will not necessarily be square.

For $r \in \mathbb{N}$, let $\omega = \omega_r$ be the primitive rth root of unity $e^{2\pi i/r}$. For $r_1 \le r_2$, let $B^{(r_2, r_1)} : \mathbb{C}^{r_1} \to \mathbb{C}^{r_2}$ be the "Walsh matrix" defined by

$$b_{ij} = \omega_{r_2}^{ij} \,, \quad i = 0, 1, \ldots, r_2 - 1 \,, \; j = 0, 1, \ldots, r_1 - 1.$$

We note the important property of this matrix: For $j_1 \ne j_2$, if we consider the complex inner product of the j_1 and j_2 column, we have

$$\sum_{i=0}^{r_2 - 1} \omega_{r_2}^{ij_1} \overline{\omega_{r_2}^{ij_2}} = \sum_{i=0}^{r_2 - 1} \omega_{r_2}^{i(j_1 - j_2)} = \frac{1 - \left(\omega_{r_2}^{j_1 - j_2}\right)^{r_2}}{1 - \omega_{r_2}^{j_1 - j_2}}$$

which equals zero, therefore the columns of $B^{(r_2,r_1)}$ are orthogonal (and have the same Euclidean norm, $r_2^{1/2}$). We see that $r_2^{-1/2}B^{(r_2,r_1)}$ can be extended to a unitary matrix, U, by including the columns (with the same definition) for $j = r_1, \ldots, r_2 - 1$. Therefore, if $v \in \mathbb{C}^{r_1}$, let $v' \in \mathbb{C}^{r_2}$ be obtained by $v' = (v_1, \ldots, v_{r_1}, 0, 0, \ldots, 0)$ and we have

$$\|B^{(r_2,r_1)}v\|^2 = r_2\|r_2^{-1/2}B^{(r_2,r_1)}v\|^2 = r_2\|Uv'\|^2 = r_2\|v'\|^2 = r_2\|v\|^2$$

(the second equality holds because the additional columns in U are multiplied by the zeroed coordinates of v'.) So, for $v \in \mathbb{C}^{r_1}$, $B^{(r_2,r_1)}$ satisfies $\|B^{(r_2,r_1)}v\|^2 = r_2\|v\|^2$.

Let $M \in \mathbb{N}$, and suppose $r_1 \leq \ldots \leq r_M$. Suppose we have M sets of complex numbers, with the jth set having r_j elements:

$$z_0^{(1)}, \ldots, z_{r_1-1}^{(1)}, z_0^{(2)}, \ldots, z_{r_2-1}^{(2)}, \ldots, z_0^{(M)}, \ldots, z_{r_M-1}^{(M)}.$$

Let $D^{(j)}$ be the $r_j \times r_j$ diagonal matrix with the diagonal entry $d_{ii} = z_i^{(j)}$. We will abbreviate

$$B^{2,1} = B^{(r_2,r_1)}, \quad B^{3,2} = B^{(r_3,r_2)}, \text{ etc.}$$

Let $u = (1, \ldots, 1) \in \mathbb{C}^{r_1}$, and consider the vector

$$D^{(M)}B^{M,M-1}D^{(M-1)} \cdots B^{3,2}D^{(2)}B^{2,1}D^{(1)}u \quad \in \mathbb{C}^{r_M} .$$

Suppose that each $z_i^{(j)}$ satisfies $|z_i^{(j)}| \leq 1$. Then we have

$$\begin{aligned}
\|D^{(M)} \cdots B^{2,1}D^{(1)}u\|^2 &\leq \|B^{M,M-1}D^{(M-1)} \cdots B^{3,2}D^{(2)}B^{2,1}D^{(1)}u\|^2 \\
&= r_M\|D^{(M-1)} \cdots B^{3,2}D^{(2)}B^{2,1}D^{(1)}u\|^2 \\
&\quad \leq \cdots \\
&= r_M \cdots r_2\|u\|^2 \\
&= r_M \cdots r_2 r_1 \\
&= \prod_1^M r_j \quad .
\end{aligned}$$

The i_M coordinate of $D^{(M)} \cdots B^{2,1}D^{(1)}u$ is

$$\sum_{i_1=0}^{r_1-1} \cdots \sum_{i_{M-1}=0}^{r_{M-1}-1} z_{i_1}^{(1)} z_{i_2}^{(2)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \omega_{r_3}^{i_2 i_3} \cdots \omega_{r_M}^{i_{M-1} i_M}.$$

The sum of the coordinates of $D^{(M)} \cdots B^{2,1} D^{(1)} u$ is less than or equal to $\left(r_M \prod_1^M r_j\right)^{1/2}$ (by the Cauchy-Schwarz inequality), and therefore we have

$$\left| \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \right| \leq \left(r_M \prod_1^M r_j \right)^{1/2}$$

when $z_i^{(j)}| \leq 1$.
We define

$$Q = Q_{r_1,\ldots,r_M} = \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M}. \tag{10}$$

Considering Q as a polynomial in the variables $z_i^{(j)}$, we have

$$\sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} |\omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M}| = \prod_1^M r_j$$

$$\|Q\|_\infty \leq \left(r_M \prod_1^M r_j \right)^{1/2}. \tag{11}$$

The important point is that the sum of the absolute values of the coefficients of Q is "large," while $\|Q\|_\infty$ is "small." This is the basic fact which allows us to construct polynomials and therefore Dirichlet series with the properties that we are interested in.

In the notation above, if we write $D^{(j)} = D_{z^{(j)}}$ and $u_1 = (1, \ldots, 1) \in \mathbb{C}^{r_1}$, $u_M = (1, \ldots, 1) \in \mathbb{C}^{r_M}$, and u_M^T is the transpose, we can also write Q as a function of the vectors $z^{(1)}, \ldots, z^{(M)}$:

$$Q(z^{(1)}, \ldots, z^{(M)}) = u^T D_{z^{(M)}} B^{M,M-1} D_{z^{(M-1)}} \cdots B^{2,1} D_{z^{(1)}} u. \tag{12}$$

Note therefore that Q is not just a polynomial in the $z_i^{(j)}$, but is in fact linear in each of the vectors $z^{(1)}, z^{(2)}, \ldots, z^{(M)}$.

4. Construction of the Dirichlet Series f

Recalling the definition of the $r_j(L)$ from equation (2), $r_1 = \lfloor 2^{\rho_1 L} \rfloor$, ..., $r_M = \lfloor 2^{\rho_M L} \rfloor$, we use equation (10) to define Q^L by

$$Q^L = Q_{r_1,\ldots,r_M} = \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M}. \tag{13}$$

When we have a polynomial Q in the complex variables $z_1, z_2, \ldots$, and $p_1, p_2, \ldots$ are primes, we can create a Dirichlet polynomial P via the substitution

$$P(s) = Q(p_1^{-s}, p_2^{-s}, \ldots) .$$

We have just defined a family of polynomials $\{Q^L\}$, each being homogenous of degree M. To create the Dirichlet polynomials and Dirichlet series that we want, we will use polynomials from this family. We define

$$P_L(s) = Q^L\left(p_{k_0^{(1)}}^{-s}, \ldots, p_{k_{r_1-1}^{(1)}}^{-s}, p_{k_0^{(2)}}^{-s}, \ldots, p_{k_{r_2-1}^{(2)}}^{-s}, \ldots, p_{k_0^{(M)}}^{-s}, \ldots, p_{k_{r_M-1}^{(M)}}^{-s}\right) . \tag{14}$$

In other words,

$$P_L = \sum_{n \in \Pi_L^\times} \gamma_n n^{-s}$$

where

$$\gamma_n = \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \qquad \text{for } n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}} \in \Pi_L^\times .$$

At this point, the idea is to consider the Dirichlet series $\sum_L \mu_L P_L$ with some coefficients μ_L. However, instead of defining the series in this way, we will define it "directly" by defining its coefficients a_n. This will be convenient since we want to consider the conditional convergence with proper care.

We will let $X > 0$ be a fixed real number which is not yet specified, but X will only depend on M and $\rho_1, \ldots, \rho_M$ (eventually, we will choose $X = \rho\frac{M+1}{2M}$ in equation (22)).

So, consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{15}$$

where

β_L is fixed but to-be-determined, with $|\beta_L| = 1$,

$$a_n = \begin{cases} \beta_L 2^{-XL} L^{-(M+2)}\, \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} & \text{if there exists an } L, \\ n \in \Pi_L^\times, \quad n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}}, \\ 0 \text{ else} . \end{cases} \tag{16}$$

Note that, because the $\Pi_L^\times$ are disjoint by equation (8), the coefficients a_n are well-defined: each $n \in \mathbb{N}$ is a member of $\Pi_L^\times$ for at most one L, and if $n \in \Pi_L^\times$ then n is given uniquely by the formula $n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}}$ for some $k_{i_1}^{(1)}, \ldots, k_{i_M}^{(M)}$ (recall that $k_i^{(j)}$ depend on L).

Having this general construction, we will now prove four bounds on the abscissae of f, in the next five sections.

The equations (15) and (16) constitute the proper definition of f. However, since the P_L have non-overlapping terms due to the disjointness of the $\Pi_L^\times$, we will use the idea

$$``f(s)=\sum_{L=1}^{\infty}\beta_L 2^{-XL}L^{-(M+2)}P_L(s)".$$

For clarity, let us formally define the "re-grouped" version of f,

$$g(s)=\sum_{L=1}^{\infty}\beta_L 2^{-XL}L^{-(M+2)}P_L(s).$$

In the region where the Dirichlet series for f converges absolutely, the above equality holds without restriction and $f=g$ since the series for f can be rearranged. For the results we are interested in, the transition from g to f is not immediate, but nevertheless our method throughout the paper will be to first prove that a result holds for g, and then to prove that it applies to f as well. To prove an upper bound on σ_b in Section 5, we first prove that the series g defines a bounded holomorphic function on a half-plane, and then we use a classic result of Bohr [5] to show that this also applies to f. To prove $\sigma_b \geq 0$ in Section 7, we reduce the question to a "finite" statement with Lemma 3, and then we work with a "grouped" series analogous to g. To prove the upper bound on σ_c in Section 9, we split a partial sum of f into two parts: a partial sum of g, and residual terms.

We begin with the two easier bounds: an upper bound on σ_b and a lower bound on σ_a.

5. An Upper Bound on the Abscissa of Boundedness

Let $\sigma>0$, let the real part of s be greater than or equal to σ, and let n_* be the smallest n in $\Pi_L^\times$,

$$n_*=p_{k_0^{(1)}}p_{k_0^{(2)}}\cdots p_{k_0^{(M)}}\ .$$

We have

$$\begin{aligned}P_L(s)&=\sum_n \gamma_n n^{-s}\\ &=n_*^{-\sigma}\sum_n \gamma_n n^{-s}n_*^{\sigma}\\ &=n_*^{-\sigma}Q^L\left(p^{\sigma}_{k_0^{(1)}}p^{-s}_{k_0^{(1)}}\ ,\ \ldots,\ p^{\sigma}_{k_0^{(1)}}p^{-s}_{k^{(1)}_{r_1-1}}\ ,\ \ldots,p^{\sigma}_{k_0^{(M)}}p^{-s}_{k_0^{(M)}}\ ,\ \ldots,p^{\sigma}_{k_0^{(M)}}p^{-s}_{k^{(M)}_{r_M-1}}\right)\end{aligned}$$

and

$$|p^{\sigma}_{k_0^{(j)}}p^{-s}_{k_i^{(j)}}|\leq 1 \quad \text{for all } i,j\ .$$

Recall (11):

$$\|Q^L\|_\infty \leq 2^{(\rho+1)L/2}\ .$$

We also have $n_* \geq c_1 2^{LM}$ by equation (6), so that

$$|P_L(s)| \leq c_1^{-\sigma} 2^{-\sigma LM} 2^{(\rho+1)L/2}$$

and therefore,

$$|\beta_L 2^{-XL} L^{-(M+2)} P_L(s)| \leq c_1^{-\sigma} 2^{-XL} 2^{-\sigma LM} 2^{(\rho+1)L/2} L^{-(M+2)}$$
$$= c_1^{-\sigma} 2^{\left[\,(1/2)(\rho+1)-\sigma M - X\,\right] L} L^{-(M+2)} \;.$$

We see that, if $(1/2)(\rho+1) - \sigma M - X < 0$, then $\sum_L \beta_L 2^{-XL} L^{-(M+2)} P_L$ defines a bounded holomorphic function in the half plane Ω_σ.

By inspection in a right half plane $\Omega_{\sigma'}$ for $\sigma' > 1$ we can conclude that

$$f = \sum_L \beta_L 2^{-XL} L^{-(M+2)} P_L$$

in $\Omega_{\sigma'}$, because the Dirichlet series for f will converge absolutely and therefore it can be rearranged to equal the right hand side. So,

$$\sum_L \beta_L 2^{-XL} L^{-(M+2)} P_L$$

gives an analytic continuation of f to a bounded function on Ω_σ, and therefore by a classic theorem of Bohr [5] we know that the Dirichlet series for f converges on Ω_σ, and f is bounded there, so $\sigma_b \leq \sigma$.

We have shown that if $\sigma > 0$ and $(1/2)(\rho+1) - \sigma M - X < 0$ then $\sigma_b \leq \sigma$. So, if

$$X \leq (1/2)(\rho+1)$$

and we choose any σ satisfying

$$\sigma > (1/2M)(\rho+1) - X/M$$

then $\sigma_b \leq \sigma$, and therefore by taking the infimum over σ we have proved the following proposition.

Proposition 1. *Let f be the Dirichlet series defined by (15) and (16). If*

$$X \leq (1/2)(\rho+1)$$

then we have

$$\sigma_b \leq (1/2M)(\rho+1) - X/M \;.$$

6. Abscissa of Absolute Convergence

We prove a lower bound on σ_a. Recall equations (6) and (7):

$$\max\{n : n \in \Pi_L^\times\} \leq c_2 2^{ML} L^M,$$
$$\left|\Pi_L^\times\right| \geq c_3 2^{\rho L}.$$

We calculate:

$$\begin{aligned}\sum |a_n| n^{-\sigma} &= \sum_{L=1}^{\infty} 2^{-XL} L^{-(M+2)} \sum_{n\in\Pi_L^\times} n^{-\sigma} \\ &\geq \sum_{L=1}^{\infty} 2^{-XL} L^{-(M+2)} \left|\Pi_L^\times\right| c_2^{-\sigma} 2^{-\sigma M L} L^{-\sigma M} \\ &\geq c_2^{-\sigma} c_3 \sum_{L=1}^{\infty} 2^{-XL} L^{-(M+2)} 2^{\rho L} 2^{-\sigma M L} L^{-\sigma M} \\ &= c_2^{-\sigma} c_3 \sum_{L=1}^{\infty} 2^{\left[\ \rho-\sigma M - X\ \right] L} L^{-\sigma M-(M+2)}\ .\end{aligned}$$

If $\rho - \sigma M - X > 0$, i.e. if

$$\sigma < (1/M)(\rho - X)$$

then the above sum is infinite, so we have the following proposition.

Proposition 2. *Let f be the Dirichlet series defined by (15) and (16). We have*

$$\sigma_a \geq (1/M)(\rho - X)\ .$$

At this point, we note that we have produced the classic Hille-Bohnenblust construction. With Propositions 1 and 2, we see that as long as we choose

$$X \leq (1/2)(\rho + 1)$$

then we have

$$\sigma_a - \sigma_b \geq \frac{1}{2M}\rho - \frac{1}{2M}.$$

For any value of M, by choosing $\rho_1 = \cdots = \rho_M = 1$ (and $X = 0$ for instance), the Dirichlet series f has terms of homogeneity exactly M and $\sigma_a - \sigma_b \geq \frac{1}{2} - \frac{1}{2M}$, the largest possible gap between σ_a and σ_b.

7. Proving $\sigma_b \geq 0$

To show that f becomes unbounded if we cross the abscissa $\sigma = 0$, i.e. to prove $\sigma_b \geq 0$, we will demonstrate that (under certain conditions on X) the partial sums of f achieve arbitrarily large values on any vertical line in the complex plane with an abscissa less than zero. This proves the bound because, if $\sigma_b < 0$ then, picking $\sigma_b < \sigma < 0$, by classic results [5] the partial sums of f converge uniformly to f on the vertical line with abscissa σ. Uniform convergence implies that there is some large N', such that

$$\text{for all } N \geq N',\ \sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n n^{-(\sigma+it)} \right| \leq 2 \sup_{t\in\mathbb{R}} |f(\sigma+it)|\ .$$

In particular, with $\sigma_b < 0$ there will be some vertical line with a negative abscissa on which the partial sums do not achieve arbitrarily large values.

We will find it easier to write the negative abscissa as $-\sigma$ with $\sigma > 0$ (instead of having σ be negative). We formalize the above discussion in the following lemma.

Lemma 3. *Let f be the Dirichlet series defined by (15) and (16). Suppose that f has the following property: for a small $\sigma > 0$ and a large $K > 0$ both arbitrary, we can find some N_K such that*

$$\sup_{t\in\mathbb{R}} \left| \sum_{N=1}^{N_K} a_n n^{-(-\sigma+it)} \right| \geq K \ .$$

If f has this property, then $\sigma_b \geq 0$.

So, let us fix $\sigma > 0$ small and $K > 0$ large. To prove that the partial sums achieve arbitrarily large values on any vertical line with a negative abscissa, we will show that the first finitely many L of the quantities $\beta_L P_L(-\sigma + it)$ attain almost total "positive interference" for some value of t: the modulus of their sum almost equals the sum of their moduli.

Our tool to show this "positive interference" is Kronecker's Theorem. As an aside, we note that this same technique is classic for proving lower bounds on the Riemann zeta function, such as in [15], chapter VIII. Note: we say that real numbers $\theta_1, \ldots, \theta_n$ are linearly independent over the integers if, for integers c_i, $\sum_1^n c_i\theta_i = 0$ implies all the c_i are zero.

Theorem 4 (Kronecker's Theorem, [2] Theorem 7.9)**.** *If $\alpha_1, \ldots, \alpha_n$ are arbitrary real numbers, $\theta_1, \ldots, \theta_n$ are real numbers that are linearly independant over the integers, and if $\epsilon > 0$, then there exists a real number t and integers $h_1, \ldots, h_n$ such that*

$$|t\theta_i - h_i - \alpha_i| < \epsilon \ , \quad \textit{for } i = 1, 2, \ldots, n.$$

Corollary 5. *For distinct primes $p_1, p_2, \ldots, p_n$, the map*

$$\overrightarrow{p} : \mathbb{R} \longrightarrow \mathbb{T}^n$$
$$\overrightarrow{p}(t) = \left(p_1^{-it}, \ldots, p_n^{-it}\right)$$

has an image that is dense in $\mathbb{T}^n$.

Proof of Corollary 5. Let $(e^{2\pi i u_1}, \ldots, e^{2\pi i u_n})$ be an arbitrary point on $\mathbb{T}^n$, and let $\epsilon > 0$. Choose $\epsilon' > 0$ such that $|e^{2\pi i x} - 1| \leq \epsilon$ for $|x| < \epsilon'$. The real numbers

$$-\log p_1/2\pi, \ \ldots, -\log p_n/2\pi$$

are linearly independent over the integers (by uniqueness of prime factorization), so by Kronecker's Theorem there exists t and integers h_j such that

$$|t\,(-\log p_j/2\pi) - h_j - u_j| < \epsilon' \,, \ \text{ for } j = 1, 2, \ldots, n$$

and so

$$\begin{aligned} |p_j^{-it} - e^{2\pi i u_j}| &= |e^{2\pi i t(-\log p_j/2\pi)} - e^{2\pi i u_j}| \\ &= |e^{2\pi i \left(t(-\log p_j/2\pi) - h_j - u_j \right)} - 1| \le \epsilon. \end{aligned}$$

□

Recall the definition of Q^L from equation (13):

$$Q^L = Q_{r_1,\ldots,r_M} = \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \quad .$$

We have already used the following map in equation (14), here we denote it by $\overrightarrow{p_L}(t)$:

$$\overrightarrow{p_L}(t) = \left(p_{k_0^{(1)}}^{-it}, \ldots, p_{k_{r_1-1}^{(1)}}^{-it}, p_{k_0^{(2)}}^{-it}, \ldots, p_{k_{r_2-1}^{(2)}}^{-it}, \ldots, p_{k_0^{(M)}}^{-it}, \ldots, p_{k_{r_M-1}^{(M)}}^{-it} \right). \quad (17)$$

We will need to consider the following polynomial, for $\sigma > 0$:

$$Q_\sigma^L(z) = \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \left[p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}} \right]^\sigma \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \ .$$

This polynomial arises because $P_L(-\sigma + it) = Q_\sigma^L\,(\overrightarrow{p_L}(t))$.

We require two lemmas, one to demonstrate "positive interference" among the $\beta_L P_L(-\sigma + it)$, and the other to estimate $\|Q_\sigma^L\|_\infty$. We will need to choose an integer $L' = L'(\sigma)$ such that

$$\text{for all } L \ge L', \ \ L^{-(M+2)} 2^{ML\sigma} \ge 1 \ \text{ and } L' \ge 2 \,. \quad (18)$$

Also, let us define the integer J_M by

$$J_M = \lceil D_M 2^M \rceil \quad (19)$$

where D_M is the constant from equation (1).

Lemma 6. *Let β_L be fixed arbitrary complex numbers of modulus one, and let $\sigma > 0$. For every large $K > 0$ there is a real t_K such that*

$$\text{for all } L \le 2L' J_M K \ \text{ we have } \left| \beta_L P_L(-\sigma + it_K) - \|Q_\sigma^L\|_\infty \right| \le (2L' J_M)^{-1}.$$

Lemma 7.

$$\|Q_\sigma^L\|_\infty \ge J_M^{-1} 2^{\rho \frac{M+1}{2M} L + ML\sigma}.$$

Proof of Lemma 6. Let us consider a fixed L for the moment. We abbreviate $n(L) = \sum_{j=1}^{M} \lfloor 2^{\rho_j L} \rfloor$, so Q_σ^L is a polynomial in $n(L)$ variables. Taking $\mathbb{T}^{n(L)}$ as the domain of Q_σ^L, we see that for any β_L, there is some point $\overrightarrow{z_L}$ in the domain of Q_σ^L such that

$$\beta_L Q_\sigma^L(\overrightarrow{z_L}) = \|Q_\sigma^L\|_\infty \ . \tag{20}$$

This is because Q_σ^L is linear in each vector $z^{(j)}$ by equation (12) (only linearity in just one of the $z^{(j)}$ is necessary). Recalling equation (17), we observe that by Corollary 5, the map

$$\begin{aligned} \overrightarrow{p_L} \ &: \mathbb{R} \to \mathbb{T}^{n(L)} \\ t &\to \overrightarrow{p_L}(t) \end{aligned}$$

has an image dense in $\mathbb{T}^{n(L)}$. Therefore, by continuity of Q_σ^L, for any ϵ we can find some t such that

$$\left|\beta_L Q_\sigma^L(\overrightarrow{p_L}(t)) - \|Q_\sigma^L\|_\infty\right| < \epsilon \ .$$

For a finite L_0, we can achieve this type of estimate for all $L \le L_0$ simultaneously. With β_L arbitrary, the map

$$\begin{aligned} \beta Q \ &: \mathbb{T}^{n(1)} \times \cdots \times \mathbb{T}^{n(L_0)} \to \mathbb{C}^{L_0} \\ (\tau_1, \ldots, \tau_{L_0}) &\to (\beta_1 Q_\sigma^1(\tau_1), \ldots, \beta_{L_0} Q_\sigma^{L_0}(\tau_{L_0})) \end{aligned}$$

is continuous. Let us consider the point $(\|Q_\sigma^1\|_\infty, \ldots, \|Q_\sigma^{L_0}\|_\infty) \in \mathbb{C}^{L_0}$. We know this point is in the image of βQ by equation (20). If we consider the ϵ-neighborhood of this point defined by

$$V = \left\{(w_1, \ldots, w_{L_0}) \ : \ \left|w_L - \|Q_\sigma^L\|_\infty\right| < \epsilon \quad \text{for all } L \le L_0\right\}$$

then, by continuity of βQ, there is an open neighborhood U in $\mathbb{T}^{n(1)} \times \cdots \times \mathbb{T}^{n(L_0)}$ with $\beta Q(U) \subset V$.

For any finite L_0, by Corollary 5 the map

$$\begin{aligned} \overrightarrow{p_1} \times \cdots \times \overrightarrow{p_{L_0}} \ &: \mathbb{R} \to \mathbb{T}^{n(1)} \times \cdots \times \mathbb{T}^{n(L_0)} \\ t &\to (\ \overrightarrow{p_1}(t),\ \ldots\ ,\ \overrightarrow{p_{L_0}}(t)\ ) \end{aligned}$$

has a dense image, so we can find $t \in \mathbb{R}$ with $(\ \overrightarrow{p_1}(t),\ \ldots\ ,\ \overrightarrow{p_{L_0}}(t)\ ) \in U$. We see that this t satisfies

$$\left|\beta_L Q_\sigma^L(\overrightarrow{p_L}(t)) - \|Q_\sigma^L\|_\infty\right| < \epsilon \quad \text{for all } L \le L_0 \ .$$

Choosing $\epsilon = (2L'J_M)^{-1}$ and $L_0 = 2L'J_M K$, and observing that $P_L(-\sigma + it) = Q_\sigma^L(\overrightarrow{p_L}(t))$, the result is proved. □

Proof of Lemma 7. Recall inequality (1) from the introduction: For any M-homogenous polynomial $\sum_{|\alpha|=M} a_\alpha z^\alpha$ in n variables, we have

$$\left(\sum_{|\alpha|=M} |a_\alpha|^{\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} \leq D_M \sup_{z\in\overline{\mathbb{D}}^n} \left|\sum_{|\alpha|=M} a_\alpha z^\alpha\right| .$$

Applying inequality (1) to Q^L_σ, and with equation (3) (and estimating $p_{k_0^{(1)}} \geq k_0^{(1)} \geq 2^L$) we see that

$$\begin{aligned}
\|Q^L_\sigma\|_\infty &\geq D_M^{-1}\left(\sum_{i_1,\ldots,i_M}\left[p_{k^{(1)}_{i_1}}\cdots p_{k^{(M)}_{i_M}}\right]^{\sigma\frac{2M}{M+1}}\right)^{\frac{M+1}{2M}} \\
&\geq D_M^{-1}\left(\sum_{i_1,\ldots,i_M} p_{k_0^{(1)}}^{(M\sigma\frac{2M}{M+1})}\right)^{\frac{M+1}{2M}} \\
&\geq D_M^{-1}\left(r_1\cdots r_M * (2^L)^{(M\sigma\frac{2M}{M+1})}\right)^{\frac{M+1}{2M}} \\
&\geq D_M^{-1}\left(r_1\cdots r_M\right)^{\frac{M+1}{2M}} 2^{ML\sigma} \\
&\geq D_M^{-1}2^{-M}2^{\rho\frac{M+1}{2M}L+ML\sigma} . \\
&\geq J_M^{-1}2^{\rho\frac{M+1}{2M}L+ML\sigma} .
\end{aligned}$$

□

Now we will prove that the hypotheses of Lemma 3 hold. Let us fix a large K and $\sigma > 0$, and let t_K be the value of t given by Lemma 6. We let $N_K = \max\{n : n \in \Pi^\times_{(2L'J_MK)}\}$, then we observe that

$$\sum_{n=1}^{N_K} a_n n^{-(-\sigma+it_K)} = \sum_{L=1}^{2L'J_MK} \beta_L 2^{-XL} L^{-(M+2)} P_L(-\sigma+it_K) .$$

We can be confident that the sum on the right hand side includes all of the terms on the left hand side (and no others) because of equation (8).

With Lemma 6, we can estimate

$$\begin{aligned}
&\left|\sum_{n=1}^{N_K} a_n n^{-(-\sigma+it_K)} - \sum_{L=1}^{2L'J_MK} 2^{-XL}L^{-(M+2)}\|Q_\sigma^L\|_\infty\right| \\
&= \left|\sum_{L=1}^{2L'J_MK} 2^{-XL}L^{-(M+2)}\Big(\beta_L P_L(-\sigma+it_K) - \|Q_\sigma^L\|_\infty\Big)\right| \\
&\le \sum_{L=1}^{2L'J_MK} 2^{-XL}L^{-(M+2)}(2L'J_M)^{-1} \\
&\le 2L'J_MK(2L'J_M)^{-1} \qquad (21)\\
&\le K\ .
\end{aligned}$$

Recalling Lemma 7, we can estimate

$$\begin{aligned}
&\sum_{L=1}^{2L'J_MK} 2^{-XL}L^{-(M+2)}\|Q_\sigma^L\|_\infty \\
&\ge \sum_{L=1}^{2L'J_MK} 2^{-XL}L^{-(M+2)}\left(J_M^{-1}2^{\rho\frac{M+1}{2M}L+ML\sigma}\right)\ .
\end{aligned}$$

We now can choose the value of X, based on the term in the exponent on the right hand side:

$$X = \rho\frac{M+1}{2M}\ . \tag{22}$$

With this, we can complete the estimate. We will drop the terms with $L < L'$ and use the properties of L' given in equation (18):

$$\begin{aligned}
\sum_{L=1}^{2L'J_MK} 2^{-XL}L^{-(M+2)}\|Q_\sigma^L\|_\infty &\ge J_M^{-1}\sum_{L=L'}^{2L'J_MK} L^{-(M+2)}2^{ML\sigma} \\
&\ge J_M^{-1}(2L'J_MK - L') \\
&\ge J_M^{-1}(2J_MK) \\
&\ge 2K.
\end{aligned}$$

This, together with equation (21), shows

$$\left|\sum_{n=1}^{N_K} a_n n^{-(-\sigma+it_K)}\right| \ge (2K) - (K) = K\ .$$

This proves that the hypotheses of Lemma 3 hold, once the choice of X is made. Applying Lemma 3 we have proved the following proposition.

Proposition 8. *Let f be the Dirichlet series defined by (15) and (16). With*

$$X = \rho \frac{M+1}{2M}$$

and for any choice of β_L, $|\beta_L| = 1$, we have $\sigma_b \geq 0$.

8. A Key Estimate

Before presenting the final bound (the upper bound on σ_c) we will require a size estimate on a sum of the following form:

$$\sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M}$$

for general P. This sum is adding terms on the unit circle with widely varying arguments, so a high degree of cancelation can be hoped for. One might expect that the size of such a sum would be roughly as large as the square root of the number of terms; here, the number of terms is roughly $2^{\rho L}$. Unfortunately, no sophisticated or impressive bound has been obtained by the current author; we will simply isolate the i_M index, sum the resulting one-variable geometric series, and then bound by absolute values. We believe that this estimate can be improved.

Lemma 9 (Key Estimate)**.**

$$\left| \sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \right| \leq c_4 2^{(\rho - \rho_{M-1})L} L \tag{23}$$

where c_4 is an absolute constant.

Note that for large M this estimates the size of the sum as only slightly smaller than the number of terms.

Proof. We fix L and P. Let us proceed with the understanding

$$n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}} \longleftrightarrow (i_1, \ldots, i_M).$$

The one key observation is that, because we selected the primes in increasing order, the set

$$\{(i_1, \ldots, i_M) : n \leq P\}$$

has the following weak convexity property: if we fix $(i_1, \ldots, i_{M-1})$, then the set

$$\{i_M : (i_1, \ldots, i_M) \text{ satisfies } n \leq P\}$$

is an "interval" of natural numbers, meaning that it equals every natural number between some (unspecified) lower and upper bounds, call them l and u, respectively. Leaving out the terms with $i_{M-1} = 0$, we have:

$$\begin{aligned}
&\sum_{n\in\Pi_L^\times, n\le P\ ,\ i_{M-1}\neq 0} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_M}^{i_{M-1}i_M} \\
&= \sum_{(i_1,\dots,i_{M-1})\ ,\ i_{M-1}\neq 0} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_{M-1}}^{i_{M-2}i_{M-1}} \sum_{i_M:(i_1,\dots,i_M)\text{ satisfies } n\le P} \omega_{r_M}^{i_{M-1}i_M} \\
&= \sum_{(i_1,\dots,i_{M-1})\ ,\ i_{M-1}\neq 0} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_{M-1}}^{i_{M-2}i_{M-1}} \sum_{i_M=l}^{u} \omega_{r_M}^{i_{M-1}i_M} \\
&= \sum_{(i_1,\dots,i_{M-1})\ ,\ i_{M-1}\neq 0} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_{M-1}}^{i_{M-2}i_{M-1}} \left(\frac{\omega_{r_M}^a - \omega_{r_M}^b}{1-\omega_{r_M}^{i_{M-1}}}\right).
\end{aligned}$$

Estimating the absolute value of this sum (and substituting j for i_{M-1}), we have

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{M-2})} \sum_{i_{M-1}=1}^{r_{M-1}-1} \frac{2}{\left|1-\omega_{r_M}^{i_{M-1}}\right|} \le \sum_{(i_1,\dots,i_{M-2})} \sum_{j=1}^{r_M-1} \frac{2}{\left|1-\omega_{r_M}^{j}\right|} \\
&\le \sum_{(i_1,\dots,i_{M-2})} 2 \sum_{1\le j\le \frac{r_M}{2}} \frac{2}{\left|1-\omega_{r_M}^{j}\right|} \\
&= 4 \sum_{(i_1,\dots,i_{M-2})} \sum_{1\le j\le \frac{r_M}{2}} \frac{1}{\left|1-e^{2\pi i j/r_M}\right|}.
\end{aligned}$$

We only sum the integer values of j between 1 and $r_M/2$. The first inequality is true because $r_{M-1} \le r_M$. For the second inequality, we note that the terms in the sum are symmetric about $r_M/2$, since $|1-\omega_{r_M}^j| = |1-\omega_{r_M}^{r_M-j}|$.

We also observe that, for $1 \le j \le r_M/2$, we have $2\pi j/r_M \in [0,\pi]$. Noting that on $[-\pi,\pi]$ there is some small $c_1 > 0$ such that $1-\cos x \ge c_1 x^2$, we have

$$\begin{aligned}
|1-e^{2\pi i j/r_M}|^2 &= 2\big(1-\cos(2\pi j/r_M)\big) \\
&\ge 2c_1(2\pi j/r_M)^2
\end{aligned}$$

and so there is an absolute constant $c_2 > 0$ such that $|1-e^{2\pi i j/r_M}| \ge c_2 j/r_M$.

Now, including those terms with $i_{M-1} = 0$, and estimating $\sum_{j=1}^{K} j^{-1} \le c_3 \log(K+1)$ with some $c_3 > 0$, we have

$$\begin{aligned}&\left|\sum_{n\in\Pi_L^\times, n\le P} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_M}^{i_{M-1}i_M}\right|\\ &\le 4c_2^{-1} \sum_{(i_1,\ldots,i_{M-2})} \sum_{1\le j\le \frac{r_M}{2}} \frac{r_M}{j} + \sum_{(i_1,\ldots,i_M):i_{M-1}=0} 1\\ &\le 4c_2^{-1} c_3\, r_1\cdots r_{M-2} r_M \big(\log(r_M/2+1)\ +1\big)\ .\end{aligned}$$

Note that $r_j \le 2^{\rho_j L}$. We have shown

$$\left|\sum_{n\in\Pi_L^\times, n\le P} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_M}^{i_{M-1}i_M}\right| \le c_4 2^{(\rho-\rho_{M-1})L} L\ .$$

□

9. Convergence on the Negative Real Axis

Note that we still have the freedom to choose β_L. We will now use β_L to arrange a large amount of cancelation in the partial sums of f at a certain point $s = -\epsilon$ (to be determined) on the negative real axis.

Fix some $\epsilon > 0$, and consider a partial sum of the series (15) at $s = -\epsilon$:

$$A_N(\epsilon) = \sum_{n=1}^{N} a_n n^\epsilon\ .$$

We define $L^*(N) = \max\{L : \text{there exists an } n \le N \text{ with } n \in \Pi_L^\times\}$. By eqution (8),

$$A_N(\epsilon) = \sum_{L<L^*(N)} \sum_{n\in\Pi_L^\times} a_n n^\epsilon + \sum_{n\in\Pi_{L^*(N)}^\times, n\le N} a_n n^\epsilon\ .$$

We would like to continue to express the inner sums in $A_N(\epsilon)$ as one-dimensional (in order to sum by parts in the desired order), so let's proceed with the understanding

$$n = p_{k_{i_1}^{(1)}}\cdots p_{k_{i_M}^{(M)}} \quad\longleftrightarrow\quad (i_1,\ldots,i_M)\ .$$

Substituting for a_n, we have

$$\begin{aligned}A_N(\epsilon) &= \sum_{L<L^*(N)} \beta_L 2^{-XL} L^{-(M+2)} \sum_{n\in\Pi_L^\times} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_M}^{i_{M-1}i_M}\ n^\epsilon\\ &+ \beta_{L^*(N)} 2^{-XL^*(N)} L^*(N)^{-(M+2)} \sum_{n\in\Pi_{L^*(N)}^\times, n\le N} \omega_{r_2}^{i_1 i_2}\cdots\omega_{r_M}^{i_{M-1}i_M} n^\epsilon\ . \qquad (24)\end{aligned}$$

Define

$$\Psi_L(\epsilon) = \sum_{n \in \Pi_L^\times} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \; n^\epsilon \; . \tag{25}$$

$$\Gamma(N, \epsilon) = \sum_{n \in \Pi_{L^*(N)}^\times, n \leq N} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \; n^\epsilon \; . \tag{26}$$

Using the generic estimate (9) for any summation by parts with the differenced quantity being positive and increasing (in this case, n^ϵ), we have essentially the same bound for $\Gamma(N,\epsilon)$ and $\Psi_L(\epsilon)$:

$$|\Psi_L(\epsilon)| \leq 2 \; (\max\{n : n \in \Pi_L^\times\})^\epsilon \max_P \left\{ \left| \sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \right| \right\}$$

and $\Gamma(N,\epsilon)$ is bounded by the same expression, with $L^*(N)$ in place of L. Recalling the bound (6) on the largest element of $\Pi_L^\times$, and using the estimate (23) from Lemma 9, we see that (with $\epsilon < 1$ assumed),

$$|\Psi_L(\epsilon)| \leq c_3 2^{\epsilon M L} L^M \; 2^{(\rho - \rho_{M-1})L} L$$

and therefore,

$$2^{-XL} L^{-(M+2)} \, |\Psi_L(\epsilon)| \leq c_3 2^{[\;(\rho - \rho_{M-1}) - X + \epsilon M \;]L} L^{-1}.$$

To arrange for the exponent on the right hand side to equal zero, we choose

$$\epsilon = \frac{-1}{M} \left[\; (\rho - \rho_{M-1}) - X \;\right] \tag{27}$$

and then we have $2^{-XL} L^{-(M+2)} \, |\Psi_L(\epsilon)| \to 0$ as $L \to \infty$. Additionally, since $2^{-XL^*(N)} L^*(N)^{-(M+2)} \Gamma(N,\epsilon)$ is bounded by the same quantity (with $L^*(N)$ substituted for L), we also have

$$2^{-XL^*(N)} L^*(N)^{-(M+2)} \Gamma(N,\epsilon) \to 0$$

as $N \to \infty$. Recall the following result from infinite series: If $a_n \geq 0$ and $a_n \to 0$, then there exists some choice of signs $d_n \in \{0,1\}$ such that the partial sums $\sum_{n \leq N} (-1)^{d_n} a_n$ converge (to some unspecified value) as $N \to \infty$. This means that there exists a choice of signs d_L such that $\sum_{L < L^*(N)} (-1)^{d_L} 2^{-XL} L^{-(M+2)} \, |\Psi_L(\epsilon)|$ converges as $N \to \infty$. We at last fix the value of β_L, so that it satisfies

$$\beta_L \Psi_L(\epsilon) = (-1)^{d_L} \, |\Psi_L(\epsilon)| \, .$$

Recalling (24), (25), (26) this means we have

$$\begin{aligned} A_N(\epsilon) = & \sum_{L < L^*(N)} (-1)^{d_L} 2^{-XL} L^{-(M+2)} \, |\Psi_L(\epsilon)| \\ & + \quad \beta_{L^*(N)} 2^{-XL^*(N)} L^*(N)^{-(M+2)} \Gamma(N,\epsilon) \; . \end{aligned}$$

We see that $A_N(\epsilon)$ converges as $N \to \infty$. So, as long as the ϵ defined in equation (27) is greater than zero, we can choose $\{\beta_L\}$ such that $f(s)$ converges at $s = -\epsilon$ on the negative real axis, and so we have proved the following proposition.

Proposition 10. *Let f be the Dirichlet series defined by (15) and (16). Then f satisfies*

$$\sigma_c \leq \frac{1}{M}\big((\rho - \rho_{M-1}) - X\big)$$

if the quantity on the right hand side is negative.

10. Results

Examining Propositions 1, 2, 8 and 10, we see that with $X = \rho\frac{M+1}{2M}$, the requirement

$$X \leq (1/2)(\rho + 1)$$

from Proposition 1 is satisfied, and so we have a Dirichlet series f which satisfies

$$\begin{aligned}
&\sigma_c \leq \frac{1}{2M^2}\big[(M-1)(\rho - \rho_{M-1}) - (M+1)\rho_{M-1}\big] \\
&\sigma_b \geq 0 \\
&\sigma_b \leq \frac{1}{2M}\left(1 - \frac{\rho}{M}\right) \\
&\sigma_a \geq \frac{M-1}{2M^2}(\rho)
\end{aligned}$$

as long as the bound on σ_c is less than zero, i.e.

$$(M-1)(\rho - \rho_{M-1}) - (M+1)\rho_{M-1} < 0 \, .$$

Proving the results stated in the introduction, for $M = 2$ and $M = 3$, is now a matter of arithmetic:

With $M = 2$, and $\rho_1, \rho_2 = 1$, we have $\sigma_b = 0$, $\sigma_a \geq 1/4$, and $\sigma_c \leq -1/4$.

With $M = 3$, we can set $\rho_2 = \rho_3 = 1$ and then

$$\begin{aligned}
&\sigma_a \geq \frac{1}{9}(\rho_1 + 2) \\
&\sigma_b \in [0, \frac{1}{18}(1 - \rho_1)] \\
&\sigma_c \leq \frac{1}{9}(\rho_1 - 1) \, .
\end{aligned}$$

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