TWO NEW PROOFS FOR THE BOUNDEDNESS OF SOLUTIONS TO $x'' + a(t)x = 0$

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ABSTRACT. In the note, two theorems are presented concerning the well-known linear differential equation $x'' + a(t)x = 0$. While the results are not new, the proofs presented simplify previous work since the Gronwall inequality is avoided which is the usual case.

In this brief note, we shall give two new and elementary proofs proving the boundedness of solutions to the well-known linear differential equation,

$$x'' + a(t)x = 0 \tag{1}$$

given various constraints on $a(\cdot)$. While the results may not be new (see[1, pp. 111-117] for some classical results) the proofs are less complex and quite general. For example, the use of differential inequalities such as Gronwall's inequality is avoided ([1, pp. 95-96]). We will show that all solutions to (1) are bounded when $a(t) > 0$ and a continuous function defined on $[0, \infty)$ with the additional requirement that $a'(t) = 0$.

Furthermore, if $a(\cdot)$ is bounded above and below by positive constants K and k, then their derivatives, too, are bounded and all solutions are stable. Moreover, the absolute values of the amplitudes form a monotonically non-increasing sequence as long as $a'(\cdot) \geq 0$.

We now state and prove our first theorem.

Theorem 1. *Given equation* (1) *where $a(\cdot)$ is in $C^1[0, \infty)$ such that $a(t) > 0$ and $a'(t) \geq 0$ then all solutions are bounded as $t \to \infty$ and the absolute values of the amplitudes form a non-increasing sequence.*

Proof. Multiply (1) by $\frac{2x'(t)}{a(t)}$, integrate from 0 to t and then integrate the first term by parts. Rearrange terms to obtain

$$\frac{x'(t)^2}{a(t)} + \int_0^t \frac{x'(s)^2 a'(s)}{a(s)^2}\, ds + x(t)^2 = x(0)^2 + \frac{x'(0)^2}{a(0)}. \tag{2}$$

Since all terms in equation (1) are positive, boundededness follows. Otherwise the left-hand side of (1) would become infinite as $t \to \infty$ while the the right-hand side remained fixed which is impossible. Next, by the Sturm Comparison Theorem ([1, pp. 114-115]]), all solutions oscillate when we compare equation (1) to the equation $x'' + a_0 x = 0$ on the interval $[0, \infty)$ and $a(0) = a_0$.

Consider now two successive critical points t_1 and t_2 where $x'(t_1) = x'(t_2) = 0$ and integrate this time from t_1 to t_2 rather than 0 to t. In that case (2) becomes,

$$\int_{t_1}^{t_2} \frac{x'(s)^2 a'(s)}{a(s)^2}\, ds + x(t_2)^2 = x(t_1)^2. \tag{3}$$

Equation (3) immediately implies that $|x(t_2)| \leq |x(t_1)|$ which proves the absolute values of the amplitudes are non-increasing. □

The second and final result concerns stability of the solutions.

Theorem 2. *Given equation* (1) *where* $a(\cdot)$ *is a continuous function on* $[0, \infty)$ *such that* $a(t) > 0$, $a'(t) \geq 0$, *and* $K > a(t) > k > 0$ *for some positive constants* K *and* k, *then all solutions are bounded as* $t \to \infty$ *and stable. Furthermore, the absolute values of the amplitudes form a non-increasing sequence.*

Proof. We need only prove stability. This follows from equation (2) since we have

$$x(t)^2 \leq x(0)^2 + \frac{x'(0)^2}{a(0)^2} \leq x(0)^2 + \frac{x'(0)^2}{k} \tag{4}$$

and

$$x'(t)^2 \leq a(t)x(0)^2 + \frac{a(t)x'(0)^2}{a(0)} \leq Kx(0)^2 + \frac{Kx'(0)^2}{k}. \tag{5}$$

Relations (4) and (5) show that given small initial conditions, both $|x(\cdot)|$ and $|x'(\cdot)|$ remain small so the solutions are indeed stable. □

Example 1. *All solutions to* $x'' + (t + \cos(t))x = 0$ *are bounded by Theorem 1.*

Example 2. *All solution to* $x'' + (2 - e^{-t})x = 0$ *are bounded and stable by Theorem 2.*

Remark. *In this note, we only needed to assume* $a'(\cdot)$ *exists, but is not necessarily continuous.*

For some further results on boundedness on this and other differential equations using Liapunov functions see [2].

References

[1] D. Sanchez, *Ordinary Differential Equations and Stability Theory: An Introduction*, Dover, New York, 1979.

[2] C. Tunç, A note on boundedness of solutions to a class or non-autonomous differential equations of second order, *Appl. Anal. Discrete Math.*, **18** (2010), 361–372.

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