A NOTE ON AN INTEGER PROGRAMMING PROBLEM THAT HAS A LINEAR PROGRAMMING SOLUTION

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Abstract. It is well-known that solutions to integer programming problems usually cannot be obtained by simply solving the corresponding linear programming relaxation. There are, however, examples of integer programming problems whose solutions can be obtained by simply solving the linear program and ignoring the integer constraints. Proving that these particular models have this trait is generally beyond the scope of a beginning course in operations research. In this paper an integer programming model, with only two constraints, is presented whose solution can be directly obtained using the standard simplex method. A proof is provided that makes a connection between analysis and operations research.

When teaching operations research, instructors make the distinction between problems that require an integer solution and those in which fractional values are acceptable. Students quickly learn that simply rounding a fractional solution to the nearest integer solution does not, in general, yield an optimal solution and can even lead to an infeasible solution. During such a course, however, examples of integer programming problems (ILP) are introduced whose solution can be found by ignoring the integrality constraints and solving the corresponding linear programming (LP) relaxation. For example, the well-known transportation, assignment, and even network flow problems can all be solved using only linear programming techniques. (See Chapters 5 and 6 of Taha [3] for more information.) Near the end of the course students learn that in general, ILPs require many more steps than the traditional simplex method, which also serves as an introduction to the concept of NP Completeness and intractability.

Although the models mentioned above do indeed yield integral solutions by LP methods, students generally do not have the opportunity through homework or other means of practice to show that LP methods yield integral solutions to certain integral models. In this paper, we provide an example of an integer programming problem that can be used to illustrate that its closed solution can be directly obtained by the simplex method. We start with some definitions.

Definition 1. *The* floor function*, denoted by* $\lfloor x \rfloor$*, is defined as the largest integer that is less than or equal to the real number* x*.*

Definition 2. *A function* f *defined on the convex set* S *is a* convex function *if, for every* x_1 *and* $x_2 \in S$ *and every* λ *such that* $0 \leq \lambda \leq 1$*,*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

Consider the following ILP:

$$\begin{aligned} \text{minimize } & z = f(1)x_1 + f(2)x_2 + \cdots + f(n)x_n \\ \text{subject to } & x_1 + 2x_2 + \cdots + nx_n = p, \\ & x_1 + x_2 + \cdots + x_n = q, \\ & x_i \geq 0 \text{ and integer, for } i = 1, \ldots, n, \end{aligned}$$

where $n \geq 2$, the parameters p and q are positive integers with $p \geq q > 0$, and f is a convex function with domain $[0, \infty)$ and range $(-\infty, \infty)$.

A particular instance of this problem was first presented by Kevin Broughan and Nan Zhu [1] who showed that the results hold for $f(x) = x^2$. We shall provide proof that the same results occur in a more general problem in which the objective function coefficients are generated by a convex function. Our approach to solving this problem is to use the simplex method to determine a closed-form solution. The coefficient matrix, representing this problem, is as follows:

x_1	x_2	x_3	$\cdots$	x_i	x_{i+1}	$\cdots$	x_n	
$f(1)$	$f(2)$	$f(3)$	$\cdots$	$f(i)$	$f(i+1)$	$\cdots$	$f(n)$	0
1	2	3	$\cdots$	i	i+1	$\cdots$	n	p
1	1	1	$\cdots$	1	1	$\cdots$	1	q

Applying elementary row operations to columns i and $i+1$, for some $i = 1, \ldots, n-1$, of the initial simplex table yields the following table:

x_1	x_2	x_3	$\cdots$	x_i	x_{i+1}	$\cdots$	x_n	
z_1	z_2	z_3	$\cdots$	z_i	z_{i+1}	$\cdots$	z_n	-z
i	$i-1$	$i-2$	$\cdots$	1	0	$\cdots$	$i-n+1$	$q(i+1)-p$
$1-i$	$2-i$	$3-i$	$\cdots$	0	1	$\cdots$	$n-i$	$p-qi$

where $z_j = -f(i)(i-j+1) - f(i+1)(j-i) + f(j)$, and $z = (q(i+1) - p)f(i) + (p-qi)f(i+1)$ for all $j = 1, \ldots, n$.

An initial basic feasible solution for our problem is readily available by choosing the value of i such that $x_i = q(i+1) - p \geq 0$ and $x_{i+1} = p - qi \geq 0$ and setting all other variables equal to zero. That is, choose i such that $i \leq p/q$ and $i + 1 \geq p/q$. If q does not divide p this is equivalent to choosing i, which must be an integer, such that $i = \lfloor p/q \rfloor$. Otherwise, $i = p/q$ or

$i = p/q - 1$. In fact, we will show that this basic feasible solution is also optimal by noting that $z_i = 0$ and $z_{i+1} = 0$ and by showing that the reduced row coefficients $z_j \geq 0$ for all $j = 1, \ldots, n$. That is, the current solution will be optimal only when

$$f(j) - f(i) \geq (j - i)(f(i + 1) - f(i))$$

holds for all nonnegative integers i and j with $i = 1, \ldots, n$, $j = 1, \ldots, n$, and $j \neq i$.

Before proving the main result of this paper, we will first prove the following lemma.

Lemma 3. *If L is an affine function and $i \leq j$, then*

$$\lambda L(j) + (1 - \lambda)L(i) = \lambda L(i) + (1 - \lambda)L(i + 1),$$

where $\lambda = 1/(1 + j - i)$.

Proof. Let $\lambda = 1/(1+j-i)$ and assume that L is the affine function defined by $L(x) = ax + b$. Now, by a simple rearrangement of terms, an equivalent expression to the above equation is given by:

$$\frac{L(j) - L(i)}{L(i + 1) - L(i)} = \frac{1 - \lambda}{\lambda}.$$

Now,

$$\frac{L(j) - L(i)}{L(i + 1) - L(i)} = \frac{(aj + b) - (ai + b)}{(a(i + 1) + b) - (ai + b)} = j - i = \frac{1 - \lambda}{\lambda}.$$

Thus, the result is established. □

Theorem 4. *Let f be a convex function with domain $[0, \infty)$ and range $(-\infty, \infty)$. The inequality*

$$f(j) - f(i) \geq (j - i)(f(i + 1) - f(i)) \tag{1}$$

holds for all nonnegative integers i and j.

Proof. **Case 1.** Assume that $i < j$. The following inequalities are equivalent to (1):

$$f(j) - f(i) \geq (j - i)f(i + 1) - (j - i)f(i),$$
$$f(j) + (j - i)f(i) \geq f(i) + (j - i)f(i + 1),$$
$$\lambda f(j) + (1 - \lambda)f(i) \geq \lambda f(i) + (1 - \lambda)f(i + 1),$$

where $\lambda = 1/(1 + j - i)$. We will prove the latter inequality.

Let L be the affine function such that $L(i) = f(i)$ and $L(j) = f(j)$. Then by Lemma 3,

$$\begin{aligned}(1-\lambda)f(i) + \lambda f(j) &= (1-\lambda)L(i) + \lambda L(j)\\ &= \lambda L(i) + (1-\lambda)L(i+1)\\ &= \lambda f(i) + (1-\lambda)L(i+1).\end{aligned}$$

Since f is convex, it follows that $L(x) \geq f(x)$ for all x in the interval $[i, j]$. In particular, $L(i+1) \geq f(i+1)$. Hence,

$$(1-\lambda)f(i) + \lambda f(j) \geq \lambda f(i) + (1-\lambda)f(i+1).$$

Case 2. Assume that $i \geq j$. The following inequalities are equivalent to (1):

$$\begin{aligned}f(j) - f(i) &\geq -(i-j)f(i+1) + (i-j)f(i),\\ f(j) + (i-j)f(i+1) &\geq (1+i-j)f(i),\\ \lambda f(j) + (1-\lambda)f(i+1) &\geq f(i),\end{aligned}$$

where $\lambda = 1/(1+i-j)$. But, this last inequality holds because of the convexity of f and the fact that $i = \lambda j + (1-\lambda)(i+1)$. □

It should be noted that the theorem above could also be proven as an immediate corollary of Lemma 16, found on page 113 of [2]. However, since the proof of Lemma 16 was left as an exercise, we have given a direct proof.

The results of Theorem 4 imply that the ILP stated above has an integer solution that can be obtained by solving the LP relaxation. Further, there is no need to use linear programming to solve particular instances of the problem since a closed form of the optimal solution to the problem is $x^*_{p/q} = q$, and $x^*_j = 0$, otherwise, with $z = f(x^*_{p/q})x^*_{p/q}$ when q divides p. When q does not divide p, the optimal solution is

$$x^*_{\lfloor \frac{p}{q} \rfloor} = q\left(\left\lfloor \frac{p}{q} \right\rfloor + 1\right) - p, \quad x^*_{\lfloor \frac{p}{q} \rfloor + 1} = p - q\left\lfloor \frac{p}{q} \right\rfloor,$$

and $x^*_j = 0$, otherwise. In this case the objective function value is

$$z = f(x^*_{\lfloor \frac{p}{q} \rfloor})x^*_{\lfloor \frac{p}{q} \rfloor} + f(x^*_{\lfloor \frac{p}{q} \rfloor + 1})x^*_{\lfloor \frac{p}{q} \rfloor + 1}.$$

This model can be used by instructors in an operations research class to illustrate that some linear programming problems always have an integer solution. In more advanced optimization classes, the problem may also be given to students, along with some timely hints, to provide them with experience in constructing a proof of this nature. The model also provides the opportunity for the instructor to emphasize the fundamental link between topics in operations research and topics in analysis.

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