PERMUTATION PATTERN AVOIDANCE AND THE CATALAN TRIANGLE

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Abstract. In the study of various objects indexed by permutations, a natural notion of minimal excluded structure, now known as a permutation pattern, has emerged and found diverse applications. One of the earliest results from the study of permutation pattern avoidance in enumerative combinatorics is that the Catalan numbers c_n count the permutations of size n that avoid any fixed pattern of size three. We refine this result by enumerating the permutations that avoid a given pattern of size three and have a given letter in the first position of their one-line notation. Since there are two parameters, we obtain triangles of numbers rather than sequences. Our main result is that there are two essentially different triangles for any of the patterns of size three, and each of these triangles generalizes the Catalan sequence in a natural way. All of our proofs are bijective, and relate the permutations being counted to recursive formulas for the triangles.

1. Introduction

A **permutation** is a bijection from a finite set to itself. The symmetric group on n letters, denoted S_n, is the group of all permutations of an n-element set $\{1, 2, \ldots, n\}$, where composition is the group operation. In this paper, we will denote a particular permutation w by its **one-line notation**,

$$w = [w_1 \ w_2 \ \cdots \ w_n],$$

where w_k is the image of $1 \leq k \leq n$ under the bijection w.

Given $w \in S_n$ and $p \in S_\ell$ with $\ell \leq n$, we say w **contains the pattern** p if there exists $k_1 < k_2 < \cdots < k_\ell$ such that $w_{k_a} < w_{k_b}$ if and only if $p_a < p_b$ for all $1 \leq a, b \leq n$. If w does not contain p, then we say w **avoids the pattern** p; equivalently, there always exists a and b with $w_{k_a} < w_{k_b}$ and $p_a > p_b$. For example, it is straightforward to check that [25143] avoids [123] as there is no triple of values that are all increasing from left to right.

Let $S_n(p) = Av_n(p)$ denote the set of all permutations in S_n that avoid a given pattern p. Then we have an integer sequence $s_n(p) := |S_n(p)|$ that counts the number of p-avoiding permutations of size n. When $s_n(p) = s_n(q)$ for all n, then we say that the permutations p and q are **Wilf equivalent**. For example, we have that $s_n([12]) = 1$ for all $n \in \mathbb{N}$ as there is a unique permutation of each size with no pair of entries increasing from left to right. Moreover, it is not hard to see that [12] and [21] are Wilf equivalent.

If we represent our permutations as matrices by placing a 1 in position (k, w_k) and 0 elsewhere, then pattern containment corresponds to containment of a sub permutation matrix. The dihedral group action on these square matrices gives rise to three **symmetry operations** that preserve Wilf equivalence: reverse, complement, and inverse. Given a permutation w, we define the **reverse** of w to be $w^r = [w_n w_{n-1} \cdots w_1]$, the **complement** of w to be $w^c = [(n+1-w_1)\ (n+1-w_2)\ \cdots\ (n+1-w_n)]$, and we let w^{-1} denote the group-theoretic inverse of w. Then, we have

$$s_n(p) = s_n(p^c) = s_n(p^r) = s_n(p^{-1})$$

since w avoids p if and only if w^c avoids p^c, and so on. For example, $[1324]^c = [4231]$, so the integer sequences $s_n([1324])$ and $s_n([4231])$ are equal.

These definitions are elementary, but have been used to study topics such as stack-sorting algorithms from computer science [6, 10], geometry of algebraic groups [1, 15], intersection cohomology [5], Mahonian statistics [4], statistical mechanics [11, 14], and various generating functions in enumerative combinatorics [2].

Simion and Schmidt [8] were among the first to consider the relationships among various permutation patterns, and they gave a bijective proof that S_3 is a single Wilf equivalence class by establishing an explicit bijection between $S_n([132])$ and $S_n([123])$. The result immediately follows because every other size three permutation is related to one of these two by a symmetry operation. The corresponding $s_n(p)$ is the Catalan sequence $c_n = \frac{1}{n+1}\binom{2n}{n}$. This sequence can also be defined recursively as

$$c_{n+1} = \sum_{k=0}^{n} c_{n-k} c_k \text{ for } n \geq 0 \text{ where } c_0 = 1. \tag{1}$$

Because this recursion conveys a very natural phenomenon that objects of size n are built from pairs of objects with complementary sizes, the Catalan numbers arise frequently in combinatorics; Stanley [9] gives over 100 objects that are counted by the Catalan numbers.

In this work, we refine the Simion-Schmidt classification by considering permutations that avoid a given pattern of size three and have a given letter in the first position of their one-line notation. That is, we let $S_n^{(k)} = \{w \in S_n : w_1 = k\}$ and define $S_n^{(k)}(p) = S_n(p) \bigcap S_n^{(k)}$. For example,

$S_4^{(2)}([123]) = \{[2143],\ [2413],\ [2431]\}$. Since there are two parameters n and k, we now have a "triangle" of numbers $s_{n,k}(p) := |S_n^{(k)}(p)|$ for each pattern p. Our main result, Theorem 2.2, is that for p of size three, there are only two essentially different triangles and each of these generalizes the Catalan sequence in a natural way. All of our proofs are bijective and relate the permutations being counted to recursive formulas for the triangles.

This first-letter refinement of S_n is a natural construction that facilitates recursive arguments: each $S_n^{(k)} \cong S_{n-1}$ by dropping the first entry and then applying the bijection from $\{1, 2, \ldots, \widehat{k}, \ldots, n\}$ to $\{1, 2, \ldots, n-1\}$, where the hat indicates omission. This results in the decomposition $S_n = \coprod_{k=1}^{n} S_n^{(k)}$. This decomposition has been used extensively for permutation pattern enumeration, in the form of generating trees introduced by West [13], and as part of the enumeration scheme approach of Vatter and Zeilberger [12]. Our work began from an attempt to understand how structures such as Bruhat order behave under this decomposition when restricted to a pattern-avoiding subset. We expect that the tools developed in enumerating these first-letter pattern classes will be helpful in such investigations.

In Section 2, we introduce some preliminary results and reduce our first-letter Wilf classification problem to determining $s_{n,k}([213])$, $s_{n,k}([123])$ and $s_{n,k}([132])$. These are proved in Sections 3, 4, and 5, respectively. We suggest some directions for future research in Section 6.

2. Catalan triangles and complements

We begin with a classical result; see [8] or [2] for a bijective proof.

Theorem 2.1. (Knuth, Simion–Schmidt) *Let c_n denote the Catalan sequence, and $s_n(p)$ denote the number of p-avoiding permutations in S_n. For any $p \in S_3$, we have $s_n(p) = c_n$ for all n.*

There are two different triangular arrays of general interest that relate to the Catalan numbers. We will distinguish the two by their shapes. We call the first the *right Catalan triangle*. This is `A009766` in the On-Line Encyclopedia of Integer Sequences [7].

$$\begin{array}{lllllll}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 2 & & & & \\
1 & 3 & 5 & 5 & & & \\
1 & 4 & 9 & 14 & 14 & & \\
1 & 5 & 14 & 28 & 42 & 42 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We denote each entry in the triangle as $c_{n,k}$, where n is the row and k is the column of the entry, each index starting from 1. Notice that the nth row of the triangle has n entries. To generate the triangle, we start with $c_{1,1} = 1$. Each successive entry is obtained by summing the entries directly above and to the left. If either of these two positions are vacant, we add zero for the corresponding position(s). Extending this recursion generates

$$c_{n,k} = \sum_{\ell=1}^{k} c_{n-1,\ell}$$

which is equivalent to the recursion $c_{n,k} = c_{n-1,k} + c_{n,k-1}$. Note that $c_{n,n}$ is the $(n-1)$st Catalan number and that the entries in row n sum to the nth Catalan number.

We call the other triangle of interest the *isosceles Catalan triangle*. This is A078391 in the On-Line Encyclopedia of Integer Sequences [7].

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 1 & & 2 & & & \\
 & & 5 & & 2 & & 2 & & 5 & & \\
 & 14 & & 5 & & 4 & & 5 & & 14 & \\
\iddots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

We denote the elements of this triangle as $t_{n,k}$. Here, n denotes the row while k indicates the position in the row, each index starting from 1. For example, $t_{5,2} = 5$. As with the right Catalan triangle, the nth row has n entries. Starting with $c_0 = c_1 = 1$, we construct the triangle by setting $t_{n,k} = c_{n-k}c_{k-1}$. Note that each $t_{n,k}$ is one of the summands from the formula (1) for the nth Catalan number. Therefore, the nth row will sum to the nth Catalan number.

We are now in a position to state our main result.

Theorem 2.2. *Let $s_{n,k}(p)$ denote the number of permutations w in S_n that avoid p as a permutation pattern and have $w_1 = k$. Then,*

$$s_{n,k}([213]) = s_{n,k}([231]) = t_{n,k}$$

and

$$s_{n,k}([123]) = s_{n,k}([132]) = s_{n,n-k+1}([312]) = s_{n,n-k+1}([321]) = c_{n,k}$$

where $t_{n,k}$ are the entries of the isosceles Catalan triangle, and $c_{n,k}$ are the entries of the right Catalan triangle.

The proof of this result will occupy the remainder of the paper. In the classification of Wilf equivalence classes for $S_n(p)$, the dihedral symmetries were a useful tool. However, only the complement symmetry is well-defined on $S_n^{(k)}(p)$.

Lemma 2.3. *Let $p \in S_3$. Then, the complement function is a bijection between $S_n^{(k)}(p)$ and $S_n^{(n-k+1)}(p^c)$.*

Proof. Since $(w^c)^c = w$, the complement is invertible. Let w be an arbitrary permutation in $S_n^{(k)}(p)$. Then, w avoids p if and only if w^c avoids p^c. Note that $w_1 = k$ by definition of $S_n^{(k)}(p)$. Since $w_1^c = n - w_1 + 1 = n - k + 1$, we have $w^c \in S_n^{(n-k+1)}(p^c)$. □

Our strategy for the proof of Theorem 2.2 will be to enumerate $s_{n,k}([213])$, $s_{n,k}([123])$ and $s_{n,k}([132])$. The remaining patterns $p \in \{[231], [321], [312]\}$ are then enumerated by Lemma 2.3.

Throughout our proofs, we will use the following auxiliary sets.

Definition 2.4. Let $w = [w_1\ w_2\ \cdots\ w_n]$ be an arbitrary permutation in $S_n^{(k)}(p)$. Then

$$w_{<k} = \{w_j \mid w_j < k\}, \qquad w_{>k} = \{w_j \mid w_j > k\},$$
$$w_{\leq k} = \{w_j \mid w_j \leq k\}, \qquad w_{\geq k} = \{w_j \mid w_j \geq k\}.$$

3. The pattern [213]

This pattern has the relation of the isosceles Catalan triangle.

Theorem 3.1. *For all n and $1 \leq k \leq n$, we have $s_{n,k}([213]) = t_{n,k}$.*

Proof. Let $w = [w_1\ w_2\ \cdots\ w_n] \in S_n^{(k)}([213])$. Since $w_1 = k$, every element of $w_{<k}$ must appear after every element of $w_{>k}$, for otherwise w does not avoid [213]. Therefore we can relabel w as

$$[k\ w_{b_1}\ w_{b_2}\ \cdots\ w_{b_{n-k}}\ w_{a_1}\ w_{a_2}\ \cdots\ w_{a_{k-1}}] \tag{2}$$

where $w_{a_j} \in w_{<k}$ and $w_{b_j} \in w_{>k}$ and the sequences (w_{a_j}) and (w_{b_j}) each avoid [213].

In fact every permutation of the form (2) whose subsequences each avoid [213] lies in $S_n^{(k)}([213])$. To see this, note that since each $w_{a_i} < w_{b_j}$, there cannot exist a [213] instance between the subsequences $(w_1) = [k]$, $(w_a) = [w_{a_1}\ w_{a_2}\ \cdots\ w_{a_{k-1}}]$, and $(w_b) = [w_{b_1}\ w_{b_2}\ \cdots\ w_{b_{n-k}}]$. Subsequences of the form $[w_{b_i}, w_{b_j}, w_{a_\ell}]$ have $w_{a_\ell} < w_b$ while subsequences of the form $[w_{b_i}, w_{a_j}, w_{a_\ell}]$ have $w_{b_i} > w_a$ so neither are [213] instances. The others are nearly identical.

There are $(k-1)$ elements in the (w_a) subsequence, and $(n-k)$ elements in the (w_b) subsequence. By Theorem 2.1, there are c_{k-1} [213]-avoiding subsequences that can be assigned to (w_a) and c_{n-k} [213]-avoiding sequences that can be assigned to (w_b), so $\left|S_n^{(k)}([213])\right| = c_{k-1}c_{n-k}$. □

4. The pattern [123]

We now proceed to classify $S_n^{(k)}([123])$. The enumeration of these sets is more complicated than of the case $p = [213]$. To aid us in this endeavor, we will define a class of functions which 'extend' a permutation in S_{n-1} beginning with i to a permutation in S_n beginning with k.

Definition 4.1. Fix n and k. Define $f : \bigcup_{i=1}^{k} S_{n-1}^{(i)} \to S_n^{(k)}$ by

$f(w) =$

$$\begin{cases} g(w) := [\quad w_1 \quad n \quad w_2 \quad \cdots \quad w_{n-1} \quad], & \text{if } i = k, \\ h(w) := [\quad k \quad w_1+\delta_1 \quad w_2+\delta_2 \quad \cdots \quad w_{n-1}+\delta_{n-1} \quad], & \text{otherwise,} \end{cases}$$

where $\delta_j = 1$ if $w_j \geq k$, and 0, otherwise.

Example 4.2. Let $w = [2431] \in S_4^{(2)}$. We can embed w into $S_5^{(2)}$ as

$$f(w) = g(w) = [w_1\ n\ w_2\ w_3\ \cdots\ w_{n-1}] = [25431].$$

We can embed w into $S_5^{(4)}$ as

$$\begin{aligned} f(w) = h(w) &= [k\ w_1+\delta_1\ w_2+\delta_2\ \cdots\ w_{n-1}+\delta_{n-1}] \\ &= [4\ (2+0)\ (4+1)\ (3+0)\ (1+0)] = [42531]. \end{aligned}$$

Lemma 4.3. *Let $w \in S_{n-1}^{(i)}$. For all $1 \leq s, t \leq n-1$, we have $w_s < w_t$ if and only if $w_s + \delta_s < w_t + \delta_t$.*

Proof. Let w_s and w_t be entries in w. Without loss of generality, assume $w_s < w_t$. Then since w_s and w_t are distinct we have

$$w_s \leq w_s + \delta_s \leq w_t \leq w_t + \delta_t.$$

If $w_s + \delta_s = w_t + \delta_t$ then we must have $\delta_s = 1$ and $\delta_t = 0$, but this implies that $k \leq w_s < w_t < k$, a contradiction. □

We now consider pattern avoidance under $f(w)$.

Lemma 4.4. *If w avoids* [123]*, then $f(w)$ avoids* [123].

Proof. Let $w = [w_1\ w_2\ \cdots\ w_{n-1}] \in S_{n-1}^{(i)}([123])$. We have that w avoids [123] if and only if for every 3-letter subsequence $1 \leq a < b < c < n$, we have $w_a > w_b$, $w_a > w_c$ or $w_b > w_c$. To prove that $f(w)$ avoids [123], we must show $g(w)$ and $h(w)$ both avoid [123].

Claim 1. $g(w)$ avoids [123].

Let $1 \leq a < b < c \leq n$ be indices for a 3-letter subsequence in $g(w)$.

Case 1.1. $g(w)_\ell = n$ for some $\ell = a, b$ or c.
For this to happen, we must have either a or b equal to 2. In the first case, our 3-letter subsequence is $[n\ w_{b-1}\ w_{c-1}]$ and in the second case, it is $[k\ n\ w_{c-1}]$. Neither are [123] instances since $w \in S_{n-1}$ so $n > w_{j-1}$ for all $2 \leq j \leq n$.

Case 1.2. $g(w)_\ell \neq n$ for $\ell = a, b$ and c.
In this case, none of a, b or c is 2, and since $g(w)_j = w_{j-1}$ for $j \neq 2$ our three letter subsequence is $[w_x\ w_y\ w_z]$ for some $1 \leq x < y < z \leq n-1$. Since w avoids [123] by assumption, this is not a [123]-instance.

Claim 2. $h(w)$ avoids [123].

Let $1 \leq a < b < c \leq n$ be indices for a 3-letter subsequence in $h(w)$. By Lemma 4.3, any subsequence in $[w_1 + \delta_1 \cdots w_{n-1} + \delta_{n-1}]$ will have the same relative ordering as in w. Since w avoids [123], there cannot be any [123] instances in $[w_1 + \delta_1 \quad \cdots \quad w_{n-1} + \delta_{n-1}]$. Thus, we need only concern ourselves with three-letter subsequences that begin with k. So, let $h(w)_a = k$.

Case 2.1. $h(w)_b \in w_{<k}$ or $h(w)_c \in w_{<k}$.
Since $h(w)_1 = h(w)_a = k$, if either $h(w)_b \in w_{<k}$ or $h(w)_c \in w_{<k}$, then $k = h(w)_a > h(w)_b$ or $k = h(w)_a > h(w)_c$. Hence, we do not have a [123]-instance.

Case 2.2. $h(w)_b \in w_{\geq k}$ and $h(w)_c \in w_{\geq k}$.
Recall that by the definition of $f(w)$ we have $w_1 = i < k$ so $h(w)_2 = w_1 + \delta_1 = w_1$. Hence, $b \geq 3$. By assumption, our 3-letter subsequence is $[k\ w_b{+}1\ w_c{+}1]$. Since w avoids [123] and $w_1 = i < k \leq w_\ell$ for $\ell = b$ or $\ell = c$, it must be that $w_b > w_c$ and therefore $h(w)_b = w_b + 1 > w_c + 1 = h(w)_c$. Hence, the 3-letter subsequence is not a [123] instance.

$\square$

We now observe that the function f is a bijection.

Lemma 4.5. *For all n and for all $k \leq n$, we have that f is a bijection of $\bigcup_{i=1}^{k} S_{n-1}^{(i)}([123])$ onto $S_n^{(k)}([123])$.*

Proof. Let $u \in S_n^{(k)}([123])$, and define $f^{-1}(u)$ by

$$f^{-1}(u) =$$

$$\begin{cases} g^{-1}(u) := [\quad u_1 \quad u_3 \quad u_4 \quad \cdots \quad u_n \quad], & \text{if } u_2 = n, \\ h^{-1}(u) := [\quad u_2 - \epsilon_2 \quad u_3 - \epsilon_3 \quad u_4 - \epsilon_4 \quad \cdots \quad u_n - \epsilon_n \quad], & \text{if } u_2 < n, \end{cases}$$

where $\epsilon_j = 1$ if $u_j > k$, and 0 otherwise.

Observe that $f(w) = g(w) = u$ only if $u_2 = n$; otherwise, $f(w) = h(w)$. Also note that $f^{-1}(u) \in S_{n-1}^{(i)}([123])$ with $i \leq k = u_1$ and $u_2 - \epsilon_2 < k$, for otherwise $u_2 - \epsilon_2 \geq k$ and so $[i\ u_2\ n]$ forms a [123] instance in u, which is a contradiction.

Hence, it suffices to show that for all $w \in \bigcup_{i=1}^{k} S_{n-1}^{(i)}([123])$ and all $u \in S_n^{(k)}([123])$, we have $g^{-1}(g(w)) = w$, $g(g^{-1}(u)) = u$, $h^{-1}(h(w)) = w$, and $h(h^{-1}(u)) = u$. These properties follow directly from the definitions as $\delta_j = \epsilon_{j+1}$ for $1 \leq j \leq n-1$. □

Now, we can calculate the number of elements in each $S_n^{(k)}([123])$ by way of recursion. In fact, the recursion developed in Lemmas 4.4 and 4.5 is the same recursion as the right Catalan Triangle recursion.

Theorem 4.6. $s_{n,k}([123]) = c_{n,k}$, *the entry in the right Catalan triangle in row n, column k for $n \geq k \geq 1$.*

Proof. Notice that $S_1^{(1)} = \{[1]\}$. Since the permutation [1] consists of only one letter, it clearly avoids [123]. Thus, $\left|S_1^{(1)}([123])\right| = 1$. Notice that $c_{1,1} = 1$ as well. By Lemmas 4.4 and 4.5,

$$\left|S_n^{(k)}([123])\right| = \left|\bigcup_{i=1}^{k} S_{n-1}^{(i)}([123])\right| = \sum_{i=1}^{k} \left|S_{n-1}^{(i)}([123])\right| = \sum_{i=1}^{k} c_{n-1,i} = c_{n,k}$$

□

5. The pattern [132]

Next, we consider $S_n^{(k)}([132])$. To enumerate these sets, we will require the following function similar to Definition 4.1.

Definition 5.1. Let $w \in S_{n-1}^{(i)}([132])$. Define $H : \bigcup_{i=1}^{k} S_{n-1}^{(i)}([132]) \to S_n^{(k)}([132])$ with $1 \leq i \leq k$ by

$$H(w) = [k \ \ w_1 + \delta_1 \ w_2 + \delta_2 \ \cdots \ w_{n-1} + \delta_{n-1}]$$

where $\delta_j = 1$ if $w_j \geq k$, and 0 otherwise. Note that this is the same function h from Definition 4.1; however, we have extended its domain to include permutations where $i = k$.

Lemma 5.2. *If w avoids* [132]*, then $H(w)$ avoids* [132]*.*

Proof. We have that w avoids [132] if and only if for every subsequence $a < b < c$, we have $w_a > w_b$, $w_a > w_c$, or $w_c > w_b$. Let $1 \leq a < b < c \leq n$ be indices for a 3-letter subsequence in $H(w)$. By Lemma 4.3, any subsequence in $[w_1 + \delta_1 \ \cdots \ w_{n-1} + \delta_{n-1}]$ will have the same relative ordering as in w. Since w avoids [132], there cannot be any [132] instances in $[w_1 + \delta_1 \ \cdots \ w_{n-1} + \delta_{n-1}]$. Thus, we need only concern ourselves with three-letter subsequences that begin with k. So let $H(w)_a = H(w)_1 = k$.

Case 2.1. $H(w)_b \in w_{<k}$ or $H(w)_c \in w_{<k}$.
Since $H(w)_a = k$, if either $H(w)_b$ or $H(w)_c$ is less than k, then our subsequence will avoid [132].

Case 2.2. $H(w)_b \in w_{>k}$ and $H(w)_c \in w_{>k}$.
By definition, we have $H(w)_2 = w_1 + \delta_1$ with $w_1 \leq k$. If $w_1 < k$, then $H(w)_2 < k$. On the other hand if $w_1 = k$, then $H(w)_2 = k + 1$. In either case, we have $H(w)_2 < H(w)_b$ and $H(w)_2 < H(w)_c$ since $H(w)_b, H(w)_c \in w_{>k}$ and all of the $H(w)_j$ are distinct. Consequently if w avoids [132], then $H(w)_b < H(w)_c$. Therefore, our subsequence is not a [132]-instance.

□

Lemma 5.3. *For all n and for all $k \leq n$, we have that H is a bijection of $\bigcup_{i=1}^{k} S_{n-1}^{(i)}([132])$ onto $S_n^{(k)}([132])$.*

Proof. The inverse of H is given by

$$H^{-1}(u) = [u_2 - \epsilon_2 \quad u_3 - \epsilon_3 \quad u_4 - \epsilon_4 \quad \cdots \ u_n - \epsilon_n],$$

where $\epsilon_j = 1$ if $w_j > k$ and 0 otherwise, just as in the proof of Lemma 4.5.

We also note that $H^{-1}(u) \in S_{n-1}^{(i)}([132])$ with $i \leq k = u_1$, for otherwise $u_2 - \epsilon_2 > k$, so $u_2 > k + 1$, and $k < u_2 < k + 1$ forms a [132] instance in u, which is a contradiction. □

Theorem 5.4. $s_{n,k}([132]) = c_{n,k}$, *the entry in the right Catalan triangle in row n, column k such that $n \geq k \geq 1$.*

Proof. Again, notice that $S_1^{(1)} = \{[1]\}$. Since the permutation [1] consists of only one letter, it clearly avoids [132]. Thus, $\left|S_1^{(1)}([132])\right| = 1$. Recall that $c_{1,1} = 1$ as well. By Lemmas 5.2 and 5.3,

$$\left|S_n^{(k)}([132])\right| = \left|\bigcup_{i=1}^{k} S_{n-1}^{(i)}([132])\right| = \sum_{i=1}^{k} \left|S_{n-1}^{(i)}([132])\right| = \sum_{i=1}^{k} c_{n-1,i} = c_{n,k}.$$

□

6. Future work

Our results suggest some directions for future research. It would be interesting to determine the number of first-letter Wilf equivalence classes in S_n for $n \geq 4$, and to see if there is a way to determine these from knowledge of the classical Wilf equivalence classes in a particular S_n. Moreover, there are notions of pattern avoidance in other Coxeter types [3], and it should be possible to generalize our results to this setting using parabolic subgroups. Finally, Vatter's enumeration schemes [12] use more general sets of pattern classes with a given prefix in order to algorithmically enumerate the full pattern class; it may be interesting to enumerate some of these sets individually.

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References

[1] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, Volume 182 of Progress in Mathematics, Birkhäuser Boston Inc., Boston, MA, 2000.

[2] M. Bóna, *Combinatorics of Permutations*, Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2004.

[3] S. Billey and A. Postnikov, *Smoothness of Schubert varieties via patterns in root subsystems*, Adv. in Appl. Math., **34.3** (2005), 447–466.

[4] E. Babson and E. Steingrímsson, *Generalized permutation patterns and a classification of the Mahonian statistics*, Sém. Lothar. Combin., **44** (2000), Art. B44b, 18 pp. (electronic).

[5] S. C. Billey and G. S. Warrington, *Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin., **13.2** (2001), 111–136.

[6] D. E. Knuth, *The Art of Computer Programming Volume 3, Sorting and Searching*, Addison-Wesley Series in Computer Science and Information Processing, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973.

[7] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[8] R. Simion and F. W. Schmidt, *Restricted permutations*, European J. Combin., **6.4** (1985), 383–406.

[9] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999.

[10] R. Tarjan, *Sorting using networks of queues and stacks*, J. Assoc. Comput. Mach., **19** (1972), 341–346.

[11] H. N. V. Temperley and E. H. Lieb, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem*, Proc. Roy. Soc. London Ser. A, **322.1549** (1971), 251–280.

[12] V. Vatter, *Enumeration schemes for restricted permutations*, Combin. Probab. Comput., **17.1** (2008), 137–159.

[13] J. West, *Generating trees and the Catalan and Schröder numbers*, Discrete Math., **146.1–3** (1995), 247–262.

[14] B. W. Westbury. *The representation theory of the Temperley-Lieb algebras*, Math. Z., **219.4** (1995), 539–565.

[15] A. Woo and A. Yong, *Governing singularities of Schubert varieties*, J. Algebra, **320.2** (2008), 495–520.

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