A COMPACT EMBEDDING FOR SEQUENCE SPACES

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ABSTRACT. The Rellich-Kondrachov Theorem is a fundamental result in the theory of Sobolev spaces. We prove an analogue of this theorem in the context of sequence spaces, using elementary methods.

1. INTRODUCTION

One of the aims of modern analysis is to characterize the relationship between various spaces of functions, for example between different types of continuous functions, or between L^p spaces. An especially important type of relationship between Banach spaces is compact embedding: we say E_1 is compactly embedded in E_2 if bounded subsets of E_1 are pre-compact subsets of E_2. Compact embeddings provide a Bolzano-Weierstrass type theorem for infinite dimensions, since a sequence that is bounded in E_1 will contain a subsequence that converges strongly in E_2. We will prove a compact embedding theorem in the context of sequence spaces using easily accessible arguments, and then explain how this result is a special case of an important compact embedding for Sobolev spaces, the Rellich-Kondrachov Theorem, whose general proof requires much more sophisticated methods [3].

We consider the ℓ^p spaces of bi-infinite sequences (a_n) of complex numbers

$$\ell^p = \left\{ (a_n) \ : \ \sum_{-\infty}^{\infty} |a_n|^p < \infty \right\}.$$

As is well-known, ℓ^p forms a Banach space with the norm $\|(a_n)\|_{\ell^p} := \left(\sum_{-\infty}^{\infty} |a_n|^p\right)^{1/p}$. We define h^1 as

$$h^1 := \left\{ (a_n) \ : \ \sum_{-\infty}^{\infty} (1+n^2)|a_n|^2 < \infty \right\},$$

with norm $\|(a_n)\|_{h^1} := \left(\sum_{-\infty}^{\infty}(1+n^2)|a_n|^2\right)^{\frac{1}{2}}$. Our goal is to show that h^1 embeds compactly in ℓ^1, i.e. bounded subsets of h^1 are pre-compact in ℓ^1. We will then relate this result to an important theorem in functional analysis, the Rellich-Kondrachov Theorem.

2. Compactness

The important ingredient in our proof is the following characterization of compactness in Banach spaces (see also Proposition 7.4 in [2]).

Proposition 2.1. *A closed bounded subset $\mathcal{K}$ in a Banach space E is compact if and only if for every $\varepsilon > 0$, there is a subspace Y_ε of E of finite dimension such that every element $x \in \mathcal{K}$ is within distance ε of Y_ε.*

Proof. Notice that a closed and bounded set in a complete metric space is compact if and only if it is totally bounded (see for example [4]), so we need only show that the given condition is equivalent to total boundedness.

Suppose that $\mathcal{K}$ is compact, and so $\mathcal{K}$ is totally bounded. Let $\varepsilon > 0$ be given. Since $\mathcal{K}$ is totally bounded, there is a finite collection of balls $B_\varepsilon(x_i)$ such that $\mathcal{K} \subseteq \cup_{i=1}^k B_\varepsilon(x_i)$. Let $Y_\varepsilon := \text{Span}(x_1, \ldots, x_k)$. Then, any $x \in \mathcal{K}$ is within distance ε of Y_ε.

Next, suppose that for any $\varepsilon > 0$, there is an appropriate finite dimensional subspace. Given $\varepsilon > 0$, consider the finite dimensional subspace $Y_{\varepsilon/3}$. Let

$$B := \left\{ y \in Y_{\varepsilon/3} \ : \ \inf_{x \in \mathcal{K}} \|x - y\| < \frac{\varepsilon}{3} \right\} \subseteq Y_{\varepsilon/3}.$$

Since $\mathcal{K}$ is bounded, so too is B. Since $Y_{\varepsilon/3}$ is finite dimensional, B is totally bounded. Thus, there exists $y_1, y_2, \ldots, y_k \in B$ such that $B \subseteq \cup_{i=1}^k B_{\varepsilon/3}(y_i)$. For each $y_i \in B$, there is an $x_i \in \mathcal{K}$ such that $\|x_i - y_i\| < \frac{\varepsilon}{3}$. Let $x \in \mathcal{K}$ be arbitrary. By definition of $Y_{\varepsilon/3}$, there is a $y \in Y_{\varepsilon/3}$ such that $\|x - y\| < \frac{\varepsilon}{3}$. Then $y \in B$ and so $y \in B_{\varepsilon/3}(y_i)$ for some i. Therefore, by the triangle inequality,

$$\|x - x_i\| \le \|x - y\| + \|y - y_i\| + \|y_i - x_i\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $x \in B_\varepsilon(x_i)$, and so $\mathcal{K} \subseteq \cup_{i=1}^k B_\varepsilon(x_i)$, which implies $\mathcal{K}$ is totally bounded. □

3. Compact Embedding

Theorem 3.1. $h^1 \subseteq \ell^1$*. Furthermore, if* $\mathcal{A} \subseteq h^1$ *is bounded in* h^1*, then* $\mathcal{A}$ *is pre-compact in* ℓ^1*.*

Proof. First, we show that $h^1 \subseteq \ell^1$. Suppose that $x = (a_n) \in h^1$. Then, by the Cauchy-Schwarz inequality, we have

$$\sum_{-\infty, n \neq 0}^{\infty} |a_n| = \sum_{-\infty, n \neq 0}^{\infty} \left(|na_n| \cdot \frac{1}{|n|} \right)$$

$$\leq \left(\sum_{-\infty}^{\infty}(1+n^2)|a_n|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{-\infty, n\neq 0}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}}.$$

Therefore, $x = (a_n) \in \ell^1$.

Next, suppose that $x = (a_n) \in \mathcal{A}$. By the inequality above, $\mathcal{A}$ is bounded in ℓ^1. In addition, by the Cauchy-Schwarz inequality, we have

$$\sum_{|n|>N} |a_n| \leq \left(\sum_{|n|>N} n^2|a_n|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{|n|>N} \frac{1}{n^2}\right)^{\frac{1}{2}}$$
$$\leq K \left(\sum_{|n|>N} \frac{1}{n^2}\right)^{\frac{1}{2}},$$

where K satisfies $\left(\sum_{|n|>N} n^2|a_n|^2\right)^{\frac{1}{2}} < K$ for all $(a_n) \in \mathcal{A}$. (Since $\mathcal{A}$ is bounded in h^1, such a K exists.) Suppose now that $\varepsilon > 0$ is given, and let N be so large that $\left(\sum_{|n|>N} \frac{1}{n^2}\right)^{\frac{1}{2}} < \frac{\varepsilon}{2K}$. We take

$$Y_\varepsilon = \mathrm{Span}(e_{-N} \ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots, e_N),$$

where e_i is the element of ℓ^p whose only non-zero entry is the ith entry, which is 1. Then, if $x = (a_n)$ (i.e. $x = \sum_{-\infty}^{\infty} a_n e_n$), taking $y = \sum_{n=-N}^{N} a_n e_n \in Y_\varepsilon$ yields

$$\|x - y\|_{\ell^1} = \sum_{|n|>N} |a_n| \leq K \left(\sum_{|n|>N} \frac{1}{n^2}\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}.$$

Thus, given an $\varepsilon > 0$, there is a subspace Y_ε of finite dimension in ℓ^1 such that every element of $\mathcal{A}$ is within $\frac{\varepsilon}{2}$ (in the ℓ^1 norm) of Y_ε. To show that $\mathcal{A}$ is pre-compact, we need to show that every element of the closure of $\mathcal{A}$ (in ℓ^1) is within ε of Y_ε. Suppose then that $x \in \overline{\mathcal{A}}$. Then, there is a sequence of $x_n \in \mathcal{A}$ such that $x_n \to x$ in ℓ^1. Then, for n sufficiently large that x_n is within $\frac{\varepsilon}{2}$ of x, the triangle inequality implies that x is within ε of Y_ε. Therefore, the closure of $\mathcal{A}$ in ℓ^1 is compact by Proposition 2.1. □

Notice that we may generalize Theorem 3.1: if $w^{1,p}$ is defined to be the set of sequences (a_n) satisfying $\sum_{n\in\mathbb{Z}}(1 + |n|^p)|a_n|^p < \infty$, then for any $p > 1$, we may use Hölder's Inequality to see that $w^{1,p} \subseteq \ell^1$, and in fact $w^{1,p}$ is embedded compactly in ℓ^1.

4. The Rellich-Kondrachov Theorem

Before stating the general version of the Rellich-Kondrachov Theorem, we include some information on Sobolev spaces. Suppose $\Omega \subseteq \mathbb{R}^k$ is bounded and open and let $C_c^\infty(\Omega)$ be the set of smooth functions with compact support in Ω. The Sobolev space $W^{1,p}(\Omega)$ is:

$$W^{1,p}(\Omega) := \Big\{ f \in L^p(\Omega) : \text{ for } i = 1, 2, \ldots, k, \text{ there exists } g_i \in L^p(\Omega) \text{ such that } \int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega g_i \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega) \Big\}.$$

The functions g_i in the definition of $W^{1,p}(\Omega)$ are often called the ith weak partial derivatives of f. Notice that if $f \in C^1(\overline{\Omega})$, then $g_i := \frac{\partial f}{\partial x_i}$ satisfies the definition above. Thus, $W^{1,p}(\Omega)$ consists of functions in $L^p(\Omega)$ whose weak derivatives are also in $L^p(\Omega)$. It can be shown that $W^{1,p}(\Omega)$ is a Banach space with the norm

$$\|f\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |f|^p + \sum_{i=1}^k |g_i|^p \, dx \right)^{\frac{1}{p}},$$

and we write $\|f\|_{W^{1,p}}$ when the domain Ω is unambiguous. Note that $W^{1,2}(\Omega)$ is a Hilbert space with inner product $(u, v) := \int_\Omega \nabla u \cdot \nabla v + u\, v \, dx$. It can be shown that $W^{1,p}(\Omega)$ is the closure of $W^{1,p}(\Omega) \cap C^1(\Omega)$ with respect to the $W^{1,p}$ norm (see [3]). Sobolev spaces play an important role in differential equations, especially in connection with the calculus of variations.

In the calculus of variations, Sobolev spaces are useful because they provide a natural setting for the weak form of the Euler-Lagrange equations. An important example: if $w \in W^{1,2}(\Omega)$ is a minimizer of $I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 \, dx$ on an open subset of $W^{1,2}(\Omega)$, then the the weak form of the Euler-Lagrange equation is $\int_\Omega \nabla w \cdot \nabla \varphi \, dx = 0$ for all $\varphi \in W^{1,2}(\Omega)$. In particular, $\int_\Omega \nabla w \cdot \nabla \varphi \, dx = 0$ for all $\varphi \in C_c^\infty(\Omega)$. This leads to Dirichlet's principle: solutions of $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$ are minimizers of $I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 \, dx$ over the set of functions in $C^2(\overline{\Omega})$ that equal f on $\partial\Omega$. Clearly, $I(u)$ is bounded from below, and so there is a sequence of functions u_n in $C^2(\overline{\Omega})$ such that

$$I(u_n) \to \inf_{v \in C^2(\overline{\Omega})} I(v)$$

and $u_n = f$ on $\partial\Omega$. However, there are no *a priori* point-wise bounds on $|\nabla u_n|$ knowing only that $\int_\Omega \frac{1}{2}|\nabla u_n|^2 \, dx$ is bounded, and hence no reason to expect a subsequence of u_n to converge pointwise. On the other hand, knowing that $\int_\Omega \frac{1}{2}|\nabla u_n|^2 \, dx$ is bounded is a step towards bounding u_n in $W^{1,2}(\Omega)$. In fact, it can be shown that u_n is bounded in $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega)$ is a Hilbert space, there is a subsequence u_{n_j} that converges weakly to some $u \in W^{1,2}(\Omega)$. In order to insure that u satisfies the boundary conditions, a stronger convergence is needed. This is where the Rellich-Kondrachov Theorem plays a role.

Theorem 4.1 (Rellich-Kondrachov). *Suppose $\Omega \subseteq \mathbb{R}^k$ is a bounded open set and the boundary of Ω is a $(k-1)$-dimensional C^1 manifold.*

1. *If $p < k$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q \in [1, p^\star)$ where $\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{k}$.*
2. *If $p = k$, then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q \geq p$.*
3. *If $p > k$, then $W^{1,p}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$.*

Since the subsequence u_{n_j} converges to u weakly in $W^{1,2}(\Omega)$, the Rellich-Kondrachov implies u_{n_j} converges to u in the $L^2(\Omega)$ norm. With this strong convergence, it can be shown that u satisfies the boundary conditions. Notice that it is still necessary to show that $u \in C^2(\overline{\Omega})$ and $I(u) = \inf_{v \in C^2(\overline{\Omega})} I(v)$. For details, see [1, Section 3.2 and Chapter 4].

5. Fourier Series

Notice that when Ω is a bounded open interval of $\mathbb{R}$, then (c) of the Rellich-Kondrachov Theorem implies $W^{1,2}((a,b))$ is compactly embedded in $C([a,b])$. We will now explain how h^1 compactly embedding in ℓ^1 implies that a subset of $W^{1,2}((-\pi,\pi))$ embeds compactly in the space of continuous 2π-periodic functions, using Fourier series. If $f \in W^{1,2}((-\pi,\pi))$, Parseval's Identity implies

$$\frac{1}{2\pi}\|f\|^2_{W^{1,2}} = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} |f|^2 + |f'|^2 \, dx\right) = \sum_{-\infty}^{\infty} \left(|a_n|^2 + |b_n|^2\right),$$

where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} \, dx$ and $b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx} \, dx$ are the Fourier coefficients of f and f'. If $f \in C^1((-\pi,\pi)) \cap W^{1,2}((-\pi,\pi))$ is 2π-periodic, integration by parts shows

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx} \, dx = -in\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx} \, dx = -ina_n,$$

and so for such f, we have

$$\frac{1}{2\pi}\|f\|^2_{W^{1,2}} = \sum_{-\infty}^{\infty} \left(1+n^2\right)|a_n|^2.$$

Thus, every $f \in C^1((-\pi,\pi)) \cap W^{1,2}((-\pi,\pi))$ that is 2π-periodic may be associated to an element of h^1. With this in mind, we define

$$W^{1,2}_{per}((-\pi,\pi)) := \left\{ f \in L^2((-\pi,\pi)) : \sum_{-\infty}^{\infty} \left(1+n^2\right) |a_n|^2 < \infty \right\},$$

where $\{a_n\}$ are the Fourier coefficients of f. To justify the notation, we now show that $W^{1,2}_{per}((-\pi,\pi)) \subseteq W^{1,2}((-\pi,\pi))$. Suppose $f \in W^{1,2}_{per}((-\pi,\pi))$. Then $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ and $g(x) := \sum_{n\in\mathbb{Z}} ina_n e^{inx}$ are both L^2 functions, where the series converge in the L^2 norm. We will show that $f \in W^{1,2}((-\pi,\pi))$, and that $f' = g$, i.e. for any $\varphi \in C_c^\infty((-\pi,\pi))$, $\int_{-\pi}^{\pi} f\varphi'\, dx = -\int_{-\pi}^{\pi} g\varphi\, dx$. For any $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} f\varphi' + g\varphi\, dx \\
&= \int_{-\pi}^{\pi} \left(f - \sum_{-N}^{N} a_n e^{inx} \right) \varphi'\, dx + \int_{-\pi}^{\pi} \left(\sum_{-N}^{N} a_n e^{inx} \right) \varphi' + g\varphi\, dx \\
&= \int_{-\pi}^{\pi} \left(f - \sum_{-N}^{N} a_n e^{inx} \right) \varphi'\, dx + \int_{-\pi}^{\pi} \left(-\sum_{-N}^{N} ina_n e^{inx} \right) \varphi + g\varphi\, dx \\
&\qquad \text{(where we have integrated by parts in the second integral)} \\
&= \int_{-\pi}^{\pi} \left(f - \sum_{-N}^{N} a_n e^{inx} \right) \varphi'\, dx + \int_{-\pi}^{\pi} \left(g - \sum_{-N}^{N} ina_n e^{inx} \right) \varphi\, dx.
\end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left| \int_{-\pi}^{\pi} f\varphi' + g\varphi\, dx \right| \\
&\le \int_{-\pi}^{\pi} \left| f - \sum_{-N}^{N} a_n e^{inx} \right| |\varphi'|\, dx + \int_{-\pi}^{\pi} \left| g - \sum_{-N}^{N} ina_n e^{inx} \right| |\varphi|\, dx \\
&\le \left\| f - \sum_{-N}^{N} a_n e^{inx} \right\|_{L^2} \|\varphi'\|_{L^2} + \left\| g - \sum_{-N}^{N} ina_n e^{inx} \right\|_{L^2} \|\varphi\|_{L^2}.
\end{aligned}$$

Letting $N \to \infty$, we see that $\int_{-\pi}^{\pi} f\varphi'\, dx = -\int_{-\pi}^{\pi} g\varphi\, dx$, as desired. In fact, this argument also shows that if $(a_n) \in h^1$, then $f(x) := \sum_{n\in\mathbb{Z}} a_n e^{inx}$ defines an element of $W^{1,2}_{per}((-\pi,\pi))$. Thus, we may identify elements of $W^{1,2}_{per}((-\pi,\pi))$ with bi-infinite sequences (a_n) for which $\sum_{n\in\mathbb{Z}}(1+n^2)|a_n|^2 <$

∞. Moreover, Parseval's Identity implies that if $f \in W^{1,2}_{per}$, then

$$\frac{1}{2\pi}\|f\|^2_{W^{1,2}_{per}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 + |f'|^2 \, dx = \sum_{n\in\mathbb{Z}} (1+n^2)|a_n|^2 = \|(a_n)\|^2_{h^1}. \quad (1)$$

Since every element of h^1 can be identified with an element of $W^{1,2}_{per}((-\pi,\pi))$ and vice versa, and (1) implies that h^1 is (up to a constant factor) isometrically isomorphic to $W^{1,2}_{per}((-\pi,\pi))$, h^1 embedding compactly in ℓ^1 will imply that $W^{1,2}_{per}((-\pi,\pi))$ is embedded compactly in an appropriate version of ℓ^1.

In one dimension, the Rellich-Kondrachov Theorem implies that $W^{1,2}((-\pi,\pi))$ embeds compactly in the space of continuous functions $C([-\pi,\pi])$. Notice that if $(a_n) \in \ell^1$, then by the Weierstrass M-test, the Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{inx}$ converges uniformly to a continuous 2π-periodic function, and so ℓ^1 can be identified with a subspace of 2π-periodic continuous functions. Therefore, the assertion $h^1 \subseteq \ell^1$ in Theorem 3.1 implies that any function in $W^{1,2}_{per}$ is a continuous 2π-periodic function. (Compare to [5], where it is shown that the Fourier series for piece-wise C^1 functions converge uniformly.) Identifying elements (a_n) of h^1 with the Fourier series of functions in $W^{1,2}_{per}$ and identifying ℓ^1 with the subspace of 2π-periodic continuous functions whose Fourier series converge absolutely, the second assertion of Theorem 3.1 implies that $W^{1,2}_{per}$ embeds compactly in the space of continuous functions, which is a version of the 1-dimensional Rellich-Kondrachov Theorem for 2π-periodic functions. (Here, we are using the fact that if $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ and $g(x) = \sum_{n\in\mathbb{Z}} b_n e^{inx}$ where $(a_n),(b_n) \in \ell^1$, then $|f(x) - g(x)| \le \sum_{n\in\mathbb{Z}} |a_n - b_n| = \|(a_n) - (b_n)\|_{\ell^1}$.)

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