WIN-LOSS SEQUENCES FOR GENERALIZED ROUNDROBIN TOURNAMENTS

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Abstract. In a tournament, each of n teams wins or loses against each of the other $n-1$ teams c times. If team i wins a total of w_i games, then the sequence $(w_1, w_2, \ldots, w_n)$ is called the *score sequence* of the tournament. In this paper we give necessary and sufficient conditions on a sequence in order that it be a score sequence for a tournament.

1. Introduction and Summary

For any given positive integers, n and c, consider a *generalized (roundrobin) tournament* $T = T_{n,c}$ in which each of n teams $1, 2, \ldots, n$ plays c games against each of the other $n-1$ teams and each game results in a win for one team and a loss for the other. If team i wins a total of w_i games, then the sequence $(w_1, w_2, \ldots, w_n)$ is called the *score sequence* of the tournament T. For any integers k and n, $0 \le k \le n$, let

$$F(k) = F(k,n) = \binom{n}{2} - \binom{n-k}{2} = \frac{k(2n-k-1)}{2}. \qquad (1.1)$$

The following result gives necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of a generalized tournament. (The necessity of the conditions follows immediately from the definitions for any c, so henceforth we shall restrict our attention to the sufficiency of the conditions.)

Theorem 1. *Let $(w_1, w_2, \ldots, w_n)$ be a sequence of n non-negative integers such that $w_1 \ge w_2 \ge \cdots \ge w_n$. Then this sequence is the score sequence of some generalized tournament $T_{n,c}$ if and only if*

$$\sum_{i=1}^{k} w_i \le cF(k) \qquad (1.2)$$

for $k = 1, 2, \ldots, n$ with equality holding when $k = n$.

Reid [10] surveys a number of proofs of this result, especially for the case $c = 1$ which dates back to Landau [8] (see also [5] and [6]). Several of the papers cited discuss procedures for constructing tournaments with a given score sequence (see also [3, p. 162]). Proofs of the general result have been given by Ford and Fulkerson [4, p. 41], Moon [9, p. 65], Bang and Sharp [1], Kemnitz and Dolff [7], and perhaps others. Most of the proofs for the general result involve showing, in effect, that if condition (1.2) holds for some given value of c, then for all i and j such that $1 \le i < j \le n$ there exist non-negative integers p_{ij} and p_{ji} such that $p_{ij} + p_{ji} = c$ and $\sum_{h \neq i} p_{ih} = w_i$. The approach followed here is rather different; it amounts to showing that the result when $c \ge 2$ follows from the result when $c = 1$. More precisely, we assume the result is known when $c = 1$ and give a direct proof of the following result (that does not require a knowledge of the integers p_{ij} and p_{ji} mentioned above; if those numbers are known, then Theorem 2 follows immediately).

Theorem 2. *Suppose the sequence $(w_1, w_2, \ldots, w_n)$ satisfies the hypothesis of Theorem 1 for some integer $c \ge 2$. Then there exist c sequences $(w_{h1}, w_{h2}, \ldots, w_{hn})$, $1 \le h \le c$, such that*

$$w_i = \sum_{h=1}^{c} w_{hi} \quad \text{for} \quad 1 \le i \le n \tag{1.3}$$

and

$$\text{each sequence } (w_{h1}, w_{h2}, \ldots, w_{hn}) \text{ is the score sequence of some ordinary tournament } T_{n,1}. \tag{1.4}$$

Consequently, if $(w_1, w_2, \ldots, w_n)$ satisfies the hypothesis of Theorem 1 for some $c \ge 2$, then the union of the c ordinary tournaments guaranteed by (1.4) produces a generalized tournament $T_{n,c}$ whose score sequence is $(w_1, w_2, \ldots, w_n)$.

We shall define the sequences $(w_{h1}, w_{h2}, \ldots, w_{hn})$ in Section 3 and establish their required properties in Section 4. First, however, we will prove some useful auxiliary results in Section 2 to avoid interrupting the flow of the argument in Section 4.

2. Auxiliary Results

In what follows n is a fixed positive integer; and, as before,

$$F(k) = \binom{n}{2} - \binom{n-k}{2} = \frac{k(2n-k-1)}{2}$$

for $0 \le k \le n$. For any given sequence of non-negative integers $(s_1, s_2, \ldots, s_n)$, let $S_0 = 0$ and

$$S_k = \sum_{i=1}^{k} s_i$$

for $1 \leq k \leq n$.

Lemma 1. *Let $(s_1, s_2, \dots, s_n)$ be a sequence of non-negative integers such that*

$$s_1 \geq \cdots \geq s_k = \cdots = s_m$$

for some integer k and m such that $1 \leq k < m \leq n$. If

$$S_{k-1} \leq F(k-1) \quad \text{and} \quad S_m \leq F(m),$$

then

$$S_k < F(h)$$

for $k \leq h < m$.

Proof. Let $e_0 = 0$ and $e_j = F(j) - S_j$ for $1 \leq j \leq m$. It follows from the definitions and assumptions that

$$\begin{aligned} s_h = s_{h+1} = S_{h+1} - S_h &= F(h+1) - F(h) + e_h - e_{h+1} \\ &= e_h - e_{h+1} + n - h - 1. \end{aligned}$$

Consequently,

$$\begin{aligned} F(h) - e_h = S_h &= S_{k-1} + (h-k+1)s_h \\ &\leq F(k-1) + (h-k+1)(e_h - e_{h+1} + n - h - 1). \end{aligned}$$

This implies, after appealing to the definitions of $F(h)$ and $F(k-1)$ and simplifying, that

$$(h-k+2)e_h \geq \binom{h-k+2}{2} + (h-k+1)e_{h+1}. \tag{2.1}$$

Now $e_m \geq 0$, by hypothesis, so $e_{m-1} \geq (m-k)/2$; and, more generally, it follows readily from (2.1) that

$$e_h \geq \frac{(m-h)(h-k+1)}{2} > 0 \tag{2.2}$$

for $k \leq h < m$. This implies the required result. (We note that the sequence $(s_1, s_2, \dots, s_{2j+1}) = (j, j, \dots, j)$ shows that inequality (2.2) is best possible, in a sense.) □

Remark. Lemma 1 give rise to the following observation: If $s_1 \geq \cdots \geq s_n$ and we want to check whether $S_h \leq F(h)$ for all h, then we needn't check those values of h such that $h < n$ and $s_h = s_{h+1}$. This observation is equivalent to a corresponding observation for sequences labeled in non-decreasing order made by Beineke and Eggleton in the 1970's but unpublished at the time; see Beineke [2, p. 49] or Reid [10, p. 180]. The argument given here may or may not be essentially the same as the argument that Beineke and Eggleton used.

For notational convenience we let I_k denote a subset of size k of some specified index set. We say that a sequence $(s_1, s_2, \ldots, s_m)$ of m nonnegative integers has property P_m if

$$\sum_{i \in I_k} s_i \leq F(k) \tag{2.3}$$

for all $1 \leq k \leq m$ and all subsets I_k of $\{1, \ldots, m\}$.

Lemma 2. *Let $(s_1, s_2, \ldots, s_m)$ denote a sequence of $m (\geq 2)$ integers such that*

$$s_i \geq s_m \geq 0 \tag{2.4}$$

for $1 \leq i \leq m-1$ and suppose the sequence $(s_1, s_2, \ldots, s_{m-1})$ has property P_{m-1}

A. *If*

$$S_m \leq F(m),$$

then the sequence $(s_1, s_2, \ldots, s_m)$ has property P_m.

B. *If*

$$S_m < F(m), \tag{2.5}$$

then the sequence $(u_1, u_2, \ldots, u_m) = (s_1, s_2, \ldots, s_m + 1)$ has property P_m.

Proof. We omit the proof of A since it is very easy. To prove B, consider any subset I_k of $\{1, \ldots, m\}$ where $1 \leq k \leq m$. We may assume that $1 \leq k \leq m-1$ and that $I_k = I_{k-1} \cup \{m\}$, where I_{k-1} is a subset of $\{1, \ldots, m-1\}$, since the required analogue of inequality (2.3) follows immediately from the assumptions in the remaining cases. We may further assume that I_{k-1} is such that the sum $\sum_{i \in I_{k-1}} s_i$ is as large as possible for the value of $k-1$ under consideration, since if the required conclusion holds with this assumption it certainly holds without the assumption. And, for notational convenience, we may also assume that $I_{k-1} = \{1, \ldots, k-1\}$ and $s_1 \geq \cdots \geq s_{m-1}$. We note that $s_m \leq s_k$, by (2.4).

Subcase 1. $s_m < s_k$. In this case let $I'_k = I_{k-1} \cup \{k\}$. Then $u_m = s_m + 1 \leq s_k$, so it follows that

$$\sum_{i \in I_k} u_i \leq \sum_{i \in I'_k} s_i \leq F(k),$$

since $(s_1, s_2, \ldots, s_{m-1})$ has property P_{m-1}.

Subcase 2. $s_m = s_k$. In this case it follows that

$$s_1 \geq \cdots \geq s_k = \cdots = s_m$$

where $1 \leq k < m$. We observe that

$$\sum_{i=1}^{k-1} s_i \leq F(k-1),$$

since $(s_1, s_2, \ldots, s_{m-1})$ has property P_{m-1}; furthermore,

$$\sum_{i=1}^{m} s_i \leq F(m) - 1 < F(m),$$

by (2.5). So it follows from Lemma 1 that, in particular,

$$\sum_{i=1}^{k} s_i < F(k).$$

Now $u_m = s_m + 1 = s_k + 1$ and $u_i = s_i$ for $1 \leq i < m$. Consequently,

$$\sum_{i \in I_k} u_i = 1 + \sum_{i=1}^{k} s_i \leq F(k),$$

as required. This suffices to complete the proof of Lemma 2. □

3. Definition of the Sequences $(w_{h1}, w_{h2}, \ldots, w_{hn})$

Let $(w_1, w_2, \ldots, w_n)$ be a sequence of n non-negative integers that satisfies the hypothesis of Theorem 1 for some $c \geq 2$. For $i = 1, \ldots, n$, let

$$w_i = c \left\lfloor \frac{w_i}{c} \right\rfloor + r_i, \tag{3.1}$$

where $0 \leq r_i < c$. Consider the c by n array in which each row consists of the numbers

$$\left\lfloor \frac{w_1}{c} \right\rfloor, \left\lfloor \frac{w_2}{c} \right\rfloor, \ldots, \left\lfloor \frac{w_n}{c} \right\rfloor.$$

If it should happen that $r_i = 0$ for all i, then $w_{hi} = \left\lfloor \frac{w_i}{c} \right\rfloor$ for all relevant values of i and h. Otherwise, let $r_a, r_b, \ldots, r_q$ be the non-zero remainders in (3.1) where $1 \leq a < b < \cdots < q \leq n$. We add $+1$ to the entries in the top r_a rows of column a; then we add $+1$ to the entries in the next r_b rows of column b, and so on, with the understanding that the "next" row after the bottom row is the top row. So as we move through the columns from left to right, we add $a+1$ to an entry in each row from top to bottom before returning to the top row. The entries in the hth row of the resulting array constitute the elements of the sequence $(w_{h1}, w_{h2}, \ldots, w_{hn})$ introduced in the statement of Theorem 2.

As an illustration of these definitions, consider the sequence

$$(w_1, w_2, w_3, w_4) = (6, 5, 5, 2);$$

this satisfies the hypothesis of Theorem 1 when $(n, c) = (4, 3)$. In this case

$$\left(\left\lfloor \frac{w_1}{c} \right\rfloor, \left\lfloor \frac{w_2}{c} \right\rfloor, \left\lfloor \frac{w_3}{c} \right\rfloor, \left\lfloor \frac{w_4}{c} \right\rfloor \right) = (2, 1, 1, 0)$$

and $r_1 = 0$ and $r_2 = r_3 = r_4 = 2$. Consequently

$$\begin{pmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \end{pmatrix} = \begin{pmatrix} 2 & 1+1 & 1+1 & 0 \\ 2 & 1+1 & 1 & 0+1 \\ 2 & 1 & 1+1 & 0+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

It is not difficult to see that each of the rows in the last array corresponds to the score sequence of an ordinary tournament with $n = 4$ and $c = 1$, as required.

The preceding verbal description of the numbers w_{hi} may be summarized more formally as follows. Let $R_1 = r_1$, $R_2 = r_1 + r_2$, ..., $R_n = r_1 + \cdots + r_n$; and let $\langle R_i \rangle$ denote the remainder when R_i is divided by c, so that $0 \leq \langle R_i \rangle < c$ for $i = 1, 2, \ldots, n$. Then

$$w_{hi} = \left\lfloor \frac{w_i}{c} \right\rfloor + \varepsilon_{hi}, \tag{3.2}$$

where

$$\varepsilon_{hi} = 1$$

if

$$\langle R_{i-1} \rangle + 1 \leq \langle R_i \rangle$$

and

$$\langle R_{i-1} \rangle + 1 \leq h \leq \langle R_i \rangle;$$

or if

$$\langle R_i \rangle < \langle R_{i-1} \rangle$$

and

$$\langle R_{i-1} \rangle + 1 \leq h \leq n \quad \text{or} \quad 1 \leq h \leq \langle R_i \rangle;$$

otherwise

$$\varepsilon_{hi} = 0.$$

4. Proof of Theorem 2

We now show that the sequences $(w_{h1}, w_{h2}, \ldots, w_{hn})$ just defined satisfy conclusions (1.3) and (1.4) for any given h, where $1 \leq h \leq c$. Since we added $+1$ to r_i of the entries in the ith column of the original array – and each of these entries was originally $\lfloor \frac{w_i}{c} \rfloor$ – it follows from (3.1) that

$$\sum_{h=1}^{c} w_{hi} = c \left\lfloor \frac{w_i}{c} \right\rfloor + r_i = w_i$$

for each i; so conclusion (1.3) holds. It remains to establish conclusion (1.4).

Assertion 1.

$$\sum_{i=1}^{k} \varepsilon_{hi} \leq \left\lfloor \frac{R_k}{c} \right\rfloor$$

for $k = 1, 2, \ldots, n$ *with equality holding when* $k = n$.

Proof. The sum considered here is the number of +1's added to entries in the first k columns of the hth row of the original array described in Section 2. Suppose $R_k = cN + q$ for a given value of k, where $0 \leq q < c$. It is not difficult to see, bearing in mind the step-by-step nature of the procedure described earlier, that if $q = 0$, then the r_k entries in the kth column that are increased by +1 are the bottom r_k entries; consequently, the sum has the same value for each row h, namely $N = R_k/c$. If, however, $q > 0$, then it is not difficult to see that the sum has the value $N = \lfloor R_k/c \rfloor$ for $q < h \leq c$ and the value $N+1 = \lfloor R_k/c \rfloor + 1$ for $1 \leq h \leq q$. So the inequality holds in any case. □

It follows from definition (3.1), the definition of R_k, and assumption (1.2) that

$$\sum_{i=1}^{k} \left(c \left\lfloor \frac{w_i}{c} \right\rfloor + r_i \right) = \sum_{i=1}^{k} c \left\lfloor \frac{w_i}{c} \right\rfloor + R_k = \sum_{i=1}^{k} w_i \leq cF(k),$$

or equivalently, that

$$\sum_{i=1}^{k} \left\lfloor \frac{w_i}{c} \right\rfloor + \frac{R_k}{c} \leq F(k), \tag{4.1}$$

for $k = 1, 2, \ldots, n$ with equality holding when $k = n$. So, in particular

$$\frac{R_n}{c} = \binom{n}{2} - \sum_{i=1}^{n} \left\lfloor \frac{w_i}{c} \right\rfloor,$$

since $F(n) = n(n-1)/2$; and, consequently, R_n/c is an integer. But this implies that equality holds in the assertion when $k = n$, in view of the observations in the preceding paragraph.

Assertion 2.

$$\sum_{i=1}^{k} w_{hi} \leq F(k)$$

for $k = 1, 2, \ldots, n$ *with equality holding when* $k = n$.

Proof. Inequality (4.1) can be written in a slightly stronger form, namely

$$\sum_{i=1}^{k} \left\lfloor \frac{w_i}{c} \right\rfloor + \left\lfloor \frac{R_k}{c} \right\rfloor \leq F(k),$$

for $k = 1, 2, \ldots, n$ with equality holding when $k = n$. So it follows from definition (3.2) and Assertion 1 that

$$\sum_{i=1}^{k} w_{hi} = \sum_{i=1}^{k} \left(\left\lfloor \frac{w_i}{c} \right\rfloor + \varepsilon_{hi} \right) \leq \sum_{i=1}^{k} \left\lfloor \frac{w_i}{c} \right\rfloor + \left\lfloor \frac{R_k}{c} \right\rfloor \leq F(k),$$

for $k = 1, 2, \ldots, n$ with equality holding when $k = n$, as required. □

If $(w_{h1}, w_{h2}, \ldots, w_{hn})$ is a non-increasing sequence, then Assertion 2 is enough to ensure that it satisfies the $c = 1$ case of condition (1.2). But, as we saw in the example in Section 3, the sequence $(w_{h1}, w_{h2}, \ldots, w_{hn})$ is not necessarily non-increasing. We need a stronger assertion to cover this possibility.

Assertion 3. *The sequence* $(w_{h1}, w_{h2}, \ldots, w_{hn})$ *has property* P_m *for* $m = 1, 2, \ldots, n$.

Proof. The conclusion certainly holds when $m = 1$, since $w_{h1} \leq F(1) = n - 1$, by Assertion 2. Now consider the sequence

$$(w_{h1}, w_{h2}, \ldots, w_{h,m-1}, \lfloor w_m/c \rfloor)$$

for some integer $m \geq 2$; we may assume, as our induction hypothesis, that the sequence $(w_{h1}, w_{h2}, \ldots, w_{h,m-1})$ has property P_{m-1}. Now

$$\begin{aligned} &\min\{w_{h1}, w_{h2}, \ldots, w_{h,m-1}, \lfloor w_m/c \rfloor\} \\ &\geq \min\left\{ \left\lfloor \frac{w_1}{c} \right\rfloor, \left\lfloor \frac{w_2}{c} \right\rfloor, \ldots, \left\lfloor \frac{w_m}{c} \right\rfloor \right\} = \left\lfloor \frac{w_m}{c} \right\rfloor, \end{aligned}$$

by the definition of the w_{hi}'s and the hypothesis that $w_1 \geq w_2 \geq \cdots \geq w_m$; so condition (2.4) of Lemma 2 is satisfied. It follows from Assertion 2 and the definition of w_{hm} that

$$\sum_{i=1}^{m-1} w_{hi} + \left\lfloor \frac{w_m}{c} \right\rfloor \leq F(m) - \varepsilon_{hi},$$

where $\varepsilon_{hi} = 0$ or 1. Consequently, if $\varepsilon_{hi} = 0$, then the sequence $(w_{h1}, w_{h2}, \ldots, w_{hm}) = (w_{h1}, w_{h2}, \ldots, w_{h,m-1}, \lfloor w_m/c \rfloor)$ has property P_m by Lemma 2A; and if $\varepsilon_{hi} = 1$, then the sequence $(w_{h1}, w_{h2}, \ldots, w_{hm}) = (w_{h1}, w_{h2}, \ldots w_{h,m-1}, \lfloor w_m/c \rfloor + 1)$ has property P_m by Lemma 2B. Hence, the assertion holds for $m = 1, 2, \ldots, n$ by induction. □

To conclude, we observe that if the elements of the sequence

$$(w_{h1}, w_{h2}, \ldots, w_{hn})$$

are relabeled in non-increasing order, then it follows from Assertion 3 that the relabeled sequence satisfies the $c = 1$ case of inequality (1.2) in Theorem 1; furthermore, equality holds when $k = n$ by Assertion 2. Hence,

$(w_{h1}, w_{h2}, \ldots, w_{hn})$ is the score sequence of some ordinary tournament $T_{n,1}$. This completes the proof of Theorem 2.

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