

ON INTEGRALS AND SUMS INVOLVING SPECIAL FUNCTIONS

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ABSTRACT. Integrals and sums involving special functions are in constant demand in applied mathematics. Rather than refer to a handbook of integrals or to a computer algebra system, we present a do-it-yourself systematic approach that shows how the evaluation of such integrals and sums can be made as simple as possible. Illustrating our method, we present several examples of integrals of Poisson type, Fourier transform, as well as integrals involving product of Bessel functions. We also obtain a new identity involving the sums of ${}_2F_1$.

1. INTRODUCTION

Integrals involving Bessel functions are of extreme importance in both mathematics and physics. The Fourier transform of radial functions on the Euclidean space \mathbb{R}^n is a typical example from Fourier analysis. For example, it is well-known that if f is a given radial function on \mathbb{R}^n , i.e., $f(x) = f_0(|x|)$ for some real valued function f_0 defined on $(0, \infty)$, then its Fourier transform is also radial and is given by the formula

$$\hat{f}(\rho) = 2\pi\rho^{1-\frac{n}{2}} \int_0^\infty r^{\frac{n}{2}} f_0(r) J_{\frac{n}{2}-1}(2\pi r\rho) dr \quad (1.1)$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of the first kind of order $(n/2) - 1$ [11]. This fact has been playing a significant role in Fourier analysis where understanding the behavior of \hat{f} is indispensable in a variety of applications. Accordingly, the problem of evaluating integrals in the form (1.1) is and will remain a permanent demand.

Anyone who has encountered integrals with special functions knows that the work involves manipulating cumbersome expressions and tricky manipulations of non-obvious sums. For a thorough discussion concerning the older methods and related results concerning integrals in the form (1.1), we refer the reader to consult Watson's monumental treatise of Bessel functions [15] and the excellent book by Luke [8]. Moreover, for those who are

interested in the use of computer algebra system (CAS), we cite, among others, [4, 5, 7, 9, 10] and [16].

The main aim of this paper is to present a systematic approach for evaluating integrals in the form (1.1). The ultimate goal is to present a do-it-yourself approach which does not assume any knowledge about specialized CAS packages. The basic idea of the presented approach is connecting the encountered integral with an initial value problem whose solution is the value of the integral. The latter offers an additional tool to examine the integral at hand. We should clarify here that it is not our aim to prove the results presented but to show how they can be obtained in a fairly straightforward manner.

In order to present our approach, we consider the class \mathcal{I} of integrals of the form

$$(F, G)_{(a,b)}(x) = \int_a^b F(t)G(xt)dt \quad (1.2)$$

where $0 \leq a < b \leq \infty$ and F and G are real valued functions defined on the real line with the property that the function $(F, G)_{(a,b)}$ has a power series representation with non-zero radius of convergence. It is clear that the class of integrals \mathcal{I} in (1.2) is more general than the class in (1.1). Examples of integrals in the class \mathcal{I} are widely available. In particular, if the given functions F and G are power series around 0 and the interval of integration $[a, b]$ lies in the interval of absolute convergence of the power series $F(t)G(xt)$, then $(F, G)_{(a,b)}$ belongs to the class \mathcal{I} .

We shall state our method in terms of a theorem whose proof follows by making use of the antiderivative procedure. Following the presentation in [17], given a function f defined on an open interval around zero and a sequence $\{a_n\}_0^\infty$, we will say that f is the ordinary power series of $\{a_n\}_0^\infty$ and write $f \leftrightarrow \{a_n\}$ if and only if $f = \sum_n a_n x^n$. Hence, if $f \leftrightarrow \{a_n\}$, then $x^{-k}(f - a_0 - a_1 x - \dots - a_{n-k-1} x^{n-k-1}) := f_{[k]} \leftrightarrow \{a_{n+k}\}$ for all positive integers k . Moreover, if $f \leftrightarrow \{a_n\}$, then $(xD_x)f \rightarrow \{na_n\}$, where $xD_x := x \frac{d}{dx}$ is the usual Euler operator. Combining the just stated two rules, it follows that if $f \leftrightarrow \{a_n\}$, then $\{n^k a_{n+l}\} \leftrightarrow (xD_x)^k f_{[l]}$ for all nonnegative integers k and l ; and more generally, if P is a polynomial then $P(xD_x)f_{[l]} \leftrightarrow \{P(n)a_{n+l}\}$.

Our main theorem is the following.

Theorem 1.1. *Let $0 \leq a < b \leq \infty$ and let F and G be real valued functions defined on the real line such that $(F, G)_{(a,b)} \in \mathcal{I}$. Let m be a nonnegative integer. For $n = 0, 1, 2, \dots$, let*

$$A_n = \frac{1}{n!} \frac{d^n}{dx^n} (F, G)_{(a,b)} \Big|_{x=0}.$$

If the sequence (A_n) satisfies a difference equation in the form

$$\sum_{l=0}^m P_l(n)A_{n+l} + \varepsilon_n = 0 \quad (1.3)$$

where each $P_l(n)$ is a polynomial and ε_n is some suitable sequence, then the function $u(x) = (F, G)_{(a,b)}(x)$ is the solution to the initial value problem

$$\begin{aligned} \sum_{l=0}^m P_l(xD_x)u_{[l]} + g &= 0 \\ (D_x)^j u \Big|_{x=0} &= A_j, \quad 0 \leq l \leq m. \end{aligned} \quad (1.4)$$

where $g \leftrightarrow \{\varepsilon_n\}$.

Obviously, the above theorem leads at least theoretically to a solution of the integral (1.2) whenever such a solution exists. In order to shed some light on this theorem and its use, we list the following few remarks.

- (i) Examples of non-trivial functions F and G where the corresponding function $(F, G)_{(a,b)}$ satisfies the assumptions of Theorem 1.1 are widely available. For instance, Theorem 1.1 can be applied to evaluate the integral (1.2) provided that the function $(F, G)_{(a,b)}$ is Γ -hypergeometric. More precisely, we say that a function

$$f: (-\infty, \infty) \rightarrow (-\infty, \infty)$$

is Γ -hypergeometric if

- (a) $f(x) = \sum_{n=0}^{\infty} a_n x^n$;
- (b) $\int_0^{\infty} x^{s-1} f(x) dx$ is a ratio of Γ -functions; and
- (c) f satisfies a Fuchsian ordinary differential equation with rational coefficients.

It is clear that if F and $G: (-\infty, \infty) \rightarrow (-\infty, \infty)$ are Γ -hypergeometric, then $(F, G)_{(-\infty, \infty)}$ given by (1.2) is also Γ -hypergeometric. Therefore, Theorem 1.1 can be applied. It should be noticed here that for the application of Theorem 1.1, it is not necessary to require that both functions F and G to be Γ -hypergeometric. An example illustrating this observation is the integral $\int_0^{\infty} t K_0(t) J_0^2(t\sqrt{x}) dt$ which will be given in Section 4, where $K_0(t)$ is the modified Bessel function of the second kind. Another example is given in Section 5.3 in which we show that the integral

$$\begin{aligned} & \int_0^{\infty} r e^{-\alpha r} K_0(\beta r) dr \\ &= \frac{1}{\alpha^2 - \beta^2} \left(\frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \ln \left[\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right] - 1 \right), \alpha, \beta > 0 \end{aligned}$$

can be obtained by solving the first order initial value problem

$$(x - x^3)u' - (2x^2 + 1)u + 1 = 0, \quad u(0) = \pi.$$

- (ii) It should be noticed here that the recursive formula (1.3) is quite important for it dictates the order of the initial value problem. Obviously, the lower the order of the polynomials, the better. Motivated by this observation, we shall classify our examples based on the type of difference equations encountered. Of course, different difference equations for the same sequence may lead to different representations of the solution function $(F, G)_{(a,b)}$. This results in obtaining (deriving) new as well as old identities. For concrete examples see the remark in Subsection 5.3.
- (iii) The use of Theorem 1.1 is not restricted to evaluate integrals. More precisely, if (A_n) is a sequence satisfying a difference equation in the form (1.3), then the function $u(x) = \sum_{n=0}^{\infty} A_n x^n$ is the solution of the initial value problem (1.4). As such, we explore a few sums some old and some new. We do so because to our knowledge, the link between sums and integrals and differential equations have not received much attention. A possible reason is that when a sum is difficult enough its associated differential equation should be even more complex and that is why perhaps not so many people wondered in that direction. And while it is indeed true that the more complicated the integral or the sum is, the harder is the differential equation, the list of examples given in the rest of this paper does show that it is a worthwhile road that can even lead to new identities. In fact, the following identity obtained in Section 6 is obtainable by solving a first order differential equation.

$$\begin{aligned} & (1-x) \sum_{n=0}^{\infty} \frac{{}_2F_1(n+\alpha, n+\alpha; n+\alpha+1; -1)}{n+\alpha} x^n \\ &= \frac{1}{\alpha} {}_2F_1(\alpha, \alpha; \alpha+1; -1) \\ & \quad - \frac{x^2 {}_2F_1(\alpha+1, 1; \alpha+2; \frac{x}{2})}{(\alpha+1)2^{\alpha+1}} - \frac{x}{\alpha 2^\alpha} \end{aligned} \tag{1.5}$$

for $-1 \leq x < 1$ where ${}_2F_1$ is the Gauss hypergeometric function. Here, we should express our thanks to Christoph Koutschan for his effort in checking this identity (using the package *HolonomicFunctions* [6]) and for his valuable feedback.

This paper is organized as follows. In Section 2, for the sake of self containment of the paper, we introduce the necessary terminology used in this paper. Sections 3, 4, and 5 are devoted to presenting various classes of

integrals in the form (1.1) which can be evaluated using the method of this paper. In particular, Section 3 is devoted to the Fourier transform of the unit ball, Section 4 presents an integral involving a product of two Bessel functions that can be obtained by solving a simple first order differential equation, and Section 5 covers various examples of integrals of Poisson type. Finally in Section 6, we shall derive some interesting old and new identities illustrating the general use of our method.

Finally, in several places in this paper, we shall interchange integration with sum. In all such places, justification is only technical and can be given by making use of the Lebesgue dominated convergence theorem. An example of how such justification can be done is given in Section 4. Since similar arguments can be used in all other places, we shall omit details.

2. PRELIMINARIES

We start this section by recalling the definitions of some special functions that we shall encounter in this paper. Let ν be any complex number. Then the Bessel function J_ν of the first kind of order ν is defined in series form by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{\nu+2n}. \quad (2.1)$$

It is well-known that J_ν is a solution of the following Bessel's differential equation:

$$x^2 u''(x) + x u'(x) + (x^2 - \nu^2) u(x) = 0. \quad (2.2)$$

The two related functions I_ν and K_ν which are given by

$$I_\nu(z) = i^{-\nu} J_\nu(iz) \quad \text{and} \quad K_\nu(z) = \frac{\pi I_{-\nu}(z) - I_\nu(z)}{2 \sin(\nu\pi)} \quad (2.3)$$

are called the modified Bessel functions of the first and the second kind, respectively.

For non-negative integers p and q , the generalized hypergeometric function is defined by the series expansion

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, \dots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k k!} x^k$$

where $(t)_k$ is the Pochhammer symbol or rising factorial which is given by

$$(t)_k = \frac{\Gamma(t+k)}{\Gamma(t)}$$

where Γ is the Gamma function. The special functions ${}_1F_1$ the confluent hypergeometric function and ${}_2F_1$ the Gauss's hypergeometric function are

the most common in literature. The functions ${}_1F_1(a; b; x)$, and ${}_2F_1(a, b; c; x)$ satisfy the following differential equations, respectively.

$$xu''(x) + (b - x)u'(x) - au(x) = 0 \tag{2.4}$$

$$x(1 - x)u''(x) + (c - (a + b + 1)x)u'(x) - abu(x) = 0. \tag{2.5}$$

A well-known function that is related to ${}_pF_q$ is the *AppellF1* function which is defined by

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n + m(\beta)m(\beta')n}{m!n!(\gamma)_n + m} x^m y^n \tag{2.6}$$

and converges absolutely for $|x| < 1$ and $|y| < 1$. The function *AppellF1* was defined by Appell in 1880 and was subsequently studied by Picard in 1881. Among the several properties enjoyed by the *AppellF1*, the following is well-known and will be used later on in the paper.

$$\frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y) = \int_0^1 \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-ux)^\beta(1-uy)^{\beta'}} du. \tag{2.7}$$

We also recall the upper and lower incomplete gamma functions, $\Gamma(a, x)$ and $\gamma(a, x)$ which are respectively given by

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt, \quad \text{and} \quad \gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt. \tag{2.8}$$

3. FOURIER TRANSFORM OF THE UNIT BALL IN \mathbb{R}^n

The Fourier transform of radial functions plays a crucial role in many branches of mathematics. In particular, it plays an important role in the study of various operators in some Banach spaces such as singular integral operators, oscillatory integral operators as well as many other applications. The basic example in this regard is the Fourier transform of the unit ball in the Euclidean space \mathbb{R}^n . Let $B(0, 1)$ be the unit ball in \mathbb{R}^n . Then it is known [12] that

$$\hat{\chi}_{B(0,1)}(\xi) = |\xi|^{-\frac{n}{2}} J_{\frac{n}{2}}(2\pi |\xi|).$$

This result is usually proved by first observing that

$$\hat{\chi}_{B(0,1)}(\xi) = 2\pi |\xi|^{-\frac{n+1}{2}} \int_0^1 J_{\frac{n}{2}-1}(2\pi |\xi| r) r^{\frac{n}{2}} dr;$$

and making use of the following identity

$$\int_0^1 J_{\mu}(ts) s^{\mu-1} (1-s^2)^{\nu} ds = t^{-\nu-1} \Gamma(\nu+1) 2^{\nu} J_{\mu+\nu+1}(t) \tag{3.1}$$

for $\mu > -1/2$, $\nu > -1$, and $t > 0$.

However, integrals in the form (3.1) can be evaluated much easier using our method. In fact, by making use of the series representation of the Bessel function J_μ , we write

$$\int_0^1 J_\mu(ts) s^{\mu-1} (1-s^2)^\nu ds = t^\mu u(t^2) \tag{3.2}$$

where $u(x) = \sum_{n=0}^\infty A_n x^n$ with A_n satisfying the difference equation

$$4(n+1)(\mu+\nu+n+2)A_{n+1} + A_n = 0.$$

Using Theorem 1.1, we conclude that u is the solution of the initial value problem

$$\begin{aligned} xu'' + (\mu + \nu + 2)u' + \frac{1}{4}u &= 0, & u(0) &= \frac{\Gamma(\nu + 1)}{2^{\mu+1}\Gamma(\mu + \nu + 2)}, \\ u'(0) &= \frac{-\Gamma(\nu + 1)}{2^{\mu+3}\Gamma(\mu + \nu + 3)}. \end{aligned}$$

Thus, we obtain

$$u(x) = 2^\nu \Gamma(\nu + 1) x^{-\frac{\mu+\nu+1}{2}} J_{\mu+\nu+1}(\sqrt{x}). \tag{3.3}$$

Hence, the identity follows by (3.2) and (3.3).

4. AN INTEGRAL INVOLVING A PRODUCT OF BESSEL FUNCTIONS

Evaluating integrals involving products of special functions is one of the main problems in mathematical analysis. In this section we shall present just one example of such type of integrals to illustrate the use of differential equations in trivializing these integrals. Consider the following integral which is also tabulated in [3]

$$\int_0^\infty t K_0(t) J_0^2(t\sqrt{x}) dt = \frac{1}{\sqrt{4x+1}}, \quad x > 0.$$

In order to derive the above integral we start by observing that

$$J_0^2(t\sqrt{x}) = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^{n+m}}{(n!)^2(m!)^2 4^{n+m}} t^{2n+2m} x^{n+m}. \tag{4.1}$$

Thus, by term-by-term integration, we obtain

$$\begin{aligned} u(x) &= \int_0^\infty t K_0(t) J_0^2(t\sqrt{x}) dt \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^{n+m} x^{n+m}}{(n!m!)^2 4^{n+m}} \int_0^\infty t^{2n+2m+1} K_0(t) dt. \end{aligned} \tag{4.2}$$

Now, we observe that

$$\int_0^\infty r^{2n+1} K_0(r) dr = 4^n (\Gamma(n+1))^2 \tag{4.3}$$

for nonnegative integer n . To see (4.3), one only needs to use Fubini's Theorem, the integral representation of K_0

$$K_0(r) = \int_0^\infty \frac{e^{-r\sqrt{s^2+1}}}{\sqrt{s^2+1}} ds$$

and the simple formula

$$\int_0^{\frac{\pi}{2}} (\cos \theta)^{2n+1} d\theta = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(n+\frac{3}{2})}.$$

Thus, by (4.2) and (4.3), we obtain

$$u(x) = \sum_{n=0}^\infty A_n x^n$$

where

$$A_n = (-1)^n \sum_{m=0}^n \binom{n}{m}^2 = (-1)^n \binom{2n}{n}.$$

Thus,

$$(n+1)A_{n+1} + 2(2n+1)A_n = 0$$

and we immediately get that $u(x)$ is the solution of the initial value problem

$$(4x+1)u'(x) + 2u(x) = 0; \quad u(0) = 1$$

which has the solution $u(x) = \frac{1}{\sqrt{4x+1}}$.

In the above calculations, we remark that the interchange of the integral and sum can be easily justified using Lebesgue dominated convergence theorem. In fact, one only needs to observe that the sequence of partial sums of (4.1) is dominated by

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \left| \frac{(-1)^{n+m}}{(n!)^2(m!)^2 4^{n+m}} t^{2n+2m} x^{n+m} \right| = I_0(t\sqrt{x})$$

and that

$$\int_0^\infty t I_0(t\sqrt{x}) |K_0(t)| dt < \infty. \tag{4.4}$$

The observation of the integrability in (4.4) is a straightforward consequence of the fact that

$$r I_0(r) K_0(r) = O(r^{-2}), \text{ as } r \rightarrow \infty$$

and

$$r I_0(r) K_0(r) = O(1), \text{ as } r \rightarrow 0^+.$$

5. INTEGRALS OF POISSON TYPE

In this section, we present as application of Theorem 1.1, closed forms for a large class of integrals in the form (1.2), which covers most of the elementary functions. In fact, we shall consider the class of integrals

$$\int_0^\infty e^{-\alpha t} g(\beta t) t^\gamma dt$$

where $g(t)$ is real analytic at 0 with a suitable power series expansion.

5.1. Functions With Direct One Step Difference Equation. We assume that $g(x) = \sum_{n=0}^\infty A_n x^n$ where (A_n) satisfies a recurrence equation of the form

$$(n^2 + \alpha_2 n + \alpha_3)A_{n+1} + (\beta_2 n + \beta_3)A_n = 0 \tag{5.1}$$

for some real $\alpha_2, \alpha_3, \beta_2$ and β_3 . When $\alpha_2 = 1 + \alpha_3$, Theorem 1.1 implies that g is a solution of the initial value problem

$$xu''(x) + (\alpha_3 + \beta_2 x)u'(x) + \beta_3 u(x) = 0, u(0) = A_0, u'(0) = A_1. \tag{5.2}$$

The following functions are examples of functions $g(x)$ that satisfy initial value problems in the form (5.2):

$$g(x) = {}_1F_1(a, b, -cx); \tag{5.3}$$

$$g(x) = x^{1-a} \Gamma(a-1, -bx) \tag{5.4}$$

for some complex numbers a, b , and c . The Poisson integrals of these functions are well-known and given by the following formula.

$$\int_0^\infty t^\gamma {}_1F_1(a, b, -\beta t) e^{-t} dt = \Gamma(\gamma + 1) {}_2F_1(\gamma + 1, a, b, -\beta); \tag{5.5}$$

$$\begin{aligned} & \int_0^\infty t^\gamma \Gamma(d-1, -\beta t) e^{-t} dt \\ &= \frac{(-\beta)^{-\gamma-1} \Gamma(d+\gamma)}{\gamma+1} {}_2F_1\left(\gamma+1, d+\gamma, 2+\gamma, \frac{1}{\beta}\right), \end{aligned} \tag{5.6}$$

where a, b , and d are complex numbers with $\text{Re}(a) > -1 - 2\gamma$, $\text{Re}(d) > -\gamma$, $\gamma > -1$, and $\beta > 0$. The integrals (5.5)-(5.6) are tabulated in both [3] and [15] and were obtained using classical methods. However, these formulas can be obtained by an application of Theorem 1.1. In particular, to prove (5.5), we replace the function ${}_1F_1$ by its power series representation and integrate term-by-term to obtain

$$\int_0^\infty e^{-t} {}_1F_1(a, b, -\beta t) t^\gamma dt = u(\beta), \tag{5.7}$$

where u is a function that is real analytic at 0 with $B_n = \frac{u^{(n)}(0)}{n!}$ satisfies the following recursive formula

$$(b + n)(n + 1)B_{n+1} + (n + \gamma + 1)(a + n)B_n = 0. \tag{5.8}$$

Thus, by applying Theorem 1.1, we get that u satisfies the initial value problem

$$\begin{aligned} x(1 + x)u'' + (b + (2 + a + \gamma)x)u' + a(1 + \gamma)u &= 0 \\ u(0) = \Gamma(\gamma + 1), \quad u'(0) = -ab^{-1}\Gamma(\gamma + 2). \end{aligned} \tag{5.9}$$

It is clear that (5.9) has the solution

$$u(x) = \Gamma(\gamma + 1) {}_2F_1(\gamma + 1, a, b, -x). \tag{5.10}$$

By (5.7) and (5.10), we obtain (5.5). By a similar argument, we can give a proof of (5.6).

It is worth pointing out that the integrals (5.5)-(5.6) can be used to derive many other interesting integrals. In particular, by considering the limit

$$\lim_{\epsilon \rightarrow 0^+} \int_0^\infty t^\gamma \Gamma(d - 1, -\beta t) e^{-\epsilon t} dt$$

and make use of (5.6), we obtain the following well-known formula

$$\int_0^\infty t^\gamma \Gamma(d - 1, -\beta t) dt = \frac{(-1)^{\gamma+1} \Gamma(d + \gamma)}{(\gamma + 1) \beta^{\gamma+1}}, \quad \text{Re}(d) > -1 \tag{5.11}$$

which is tabulated in [3] and [15]. For more examples on a large class of integrals that can be derived from (5.5)-(5.6) we refer readers to [3] and [15].

5.2. Functions With Indirect One Step Difference Equation. It is clear that (5.5)-(5.6) give the Poisson integrals of functions in the form $t^l g(t)$ where $A_n = \frac{g^{(n)}(0)}{n!}$ satisfies a difference equation in the form (5.1). However, there are many functions $g(t)$ with power series coefficients that do not obey a difference equation with one step difference but still can be dealt with. In particular, we consider functions in the form $g(t) = h(t^2)$ where $\frac{g^{(n)}(0)}{n!}$ does not satisfy a one step difference equation but $\frac{h^{(n)}(0)}{n!}$ does. Because of their practical importance, we choose the example of the function $t^{-\mu} J_\nu(\beta t)$.

5.2.1. Poisson Kernel. Poisson kernels are integral kernels that have applications in various fields such as potential theory, complex analysis, theory of harmonic functions, and control theory. The importance of Poisson kernels is not limited to their use as tools in the various fields but lies in the huge amount of research they have generated. In particular, they have been the

main tools in solving the two dimensional Laplace equations with Dirichlet boundary conditions, characterizing Hardy spaces both on the unit disk and in the space, characterizing the Abel means in the theory of Fourier series and studying singular integral operators on Lebesgue spaces to name a few applications. The well-known Poisson kernel on the upper half space is given by

$$\hat{f}_\alpha(t) = \Gamma((n+1)/2)\alpha\pi^{-(n+1)/2}(\alpha^2 + |t|^2)^{-(n+1)/2} \quad (5.12)$$

for $\alpha > 0$. The classical derivation of the formula (5.12) involves very technical arguments and calculations [11]. However, Theorem 1.1 can be used to give a simple derivation of the formula (5.12). In fact, we consider the radial function $f_\alpha(x) = e^{-2\pi\alpha|x|}$, $x \in \mathbb{R}^n$ which by (1.1) can be easily shown to have the Fourier transform

$$\hat{f}_\alpha(t) = 2\pi|t|^{-\frac{n+2}{2}} \int_0^\infty r^{\frac{n}{2}} e^{-2\pi\alpha r} J_{\frac{n}{2}-1}(2\pi|t|r) dr, \quad t \in \mathbb{R}^n.$$

Using the series representation of the Bessel function, we obtain

$$\begin{aligned} \hat{f}_\alpha(t) &= \frac{2\pi|t|^{-\frac{n+2}{2}}}{(2\pi\alpha)^{\frac{n}{2}+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\frac{n}{2}-1+m+1)} \left(\frac{|t|}{2\alpha}\right)^{\frac{n}{2}-1+2m} \int_0^\infty r^{n+2m-1} e^{-r} dr \\ &= \frac{1}{2^{n-1}\alpha^n\pi^{\frac{n}{2}}} \sum_{m=0}^{\infty} \frac{\Gamma(n+2m)}{m!\Gamma(\frac{n}{2}+m)} \left(-\frac{|t|^2}{4\alpha^2}\right)^m, \end{aligned} \quad (5.13)$$

Therefore,

$$\hat{f}_\alpha(t) = \frac{1}{2^{n-1}\alpha^n\pi^{\frac{n}{2}}} u\left(\frac{-|t|^2}{4\alpha^2}\right) \quad (5.14)$$

where $u(x) = \sum_{m=0}^{\infty} A_m x^m$ and

$$A_m = \frac{\Gamma(n+2m)}{m!\Gamma(\frac{n}{2}+m)}.$$

It is easy to see that the sequence A_m satisfies the equation

$$\begin{aligned} &\left[(m+1)^2 + \left(\frac{n}{2}-1\right)(m+1)\right] A_{m+1} \\ &\quad - [4m^2 + (4n+2)m + n(n+1)] A_m = 0. \end{aligned}$$

By Theorem 1.1, we obtain that u is the solution of the initial value problem

$$\begin{aligned} x(1-4x)u'' + \left(\frac{n}{2} - (6+4n)x\right)u' + n(n+1)u &= 0, \\ u(0) = 2^{n-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right), \quad u'(0) &= 2^{n+1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{n+3}{2}\right). \end{aligned}$$

Therefore,

$$u(x) = 2^{n-1} \pi^{-\frac{1}{2}} {}_2F_1 \left(\frac{n}{2}, \frac{n+1}{2}, \frac{n}{2}, -4x \right);$$

which when combined with (5.14) imply (5.12).

Following a similar argument as above, we can derive the following generalization of the integral $\hat{f}_\alpha(t)$ which can also be found in [15]:

$$\begin{aligned} & \int_0^\infty t^\gamma J_\nu(\beta t) e^{-\alpha t} dt \\ &= \frac{\beta^\nu \Gamma(\nu + \gamma + 1)}{2^\nu \alpha^{\nu + \gamma + 1} \Gamma(\nu + 1)} {}_2F_1 \left(\frac{\nu + \gamma + 1}{2}, \frac{\nu + \gamma + 2}{2}, \nu + 1, -\frac{\beta^2}{\alpha^2} \right) \end{aligned} \quad (5.15)$$

for $\alpha > 0$, $\gamma > -1$ and complex numbers β and ν with $\operatorname{Re}(\nu) > -1$. When $\nu = \gamma = 0$, the integral (5.15) leads to the Lipschitz integral (see [15, p. 384]).

In addition, using limiting arguments and (5.15), one can derive several well-known integrals such as the following integral tabulated in [3, p. 692]

$$\int_0^\infty J_\nu(br) dr = b^{-1} \quad (\operatorname{Re}(\nu) > -1, b > 0),$$

and the Weber integral

$$\begin{aligned} \lim_{a \rightarrow 0^+} \int_0^\infty r^{-\nu + \mu - 1} e^{-ar} J_\mu(r) dr &= \int_0^\infty r^{-\nu + \mu - 1} J_\mu(r) dr \\ &= \frac{\Gamma(\frac{\mu}{2})}{2^{\nu - \mu + 1} \Gamma(\nu - \frac{\mu}{2} + 1)}, \end{aligned}$$

where $0 < \operatorname{Re}(\mu) < \operatorname{Re}(\nu) + \frac{1}{2}$.

Among several other consequences of (5.15), we also have [3, 15]:

$$\int_0^\infty t^\gamma e^{-\alpha t} \sin \beta t dt = {}_2F_1 \left(\frac{\gamma + 2}{2}, \frac{\gamma + 3}{2}, \frac{3}{2}, -\frac{\beta^2}{\alpha^2} \right) \frac{\beta \Gamma(\gamma + 2)}{\alpha^{\gamma + 2}}; \quad (5.16)$$

$$\int_0^\infty t^\gamma e^{-\alpha t} \cos \beta t dt = {}_2F_1 \left(\frac{\gamma + 1}{2}, \frac{\gamma + 2}{2}, \frac{1}{2}, -\frac{\beta^2}{\alpha^2} \right) \frac{\Gamma(\gamma + 1)}{\alpha^{\gamma + 1}}, \quad (5.17)$$

where $\alpha, \beta > 0$ and $\gamma > -1$.

5.3. Functions With Two Step Difference Equation. In this section, we discuss Poisson integrals of functions leading to difference equations with two step difference.

5.3.1. *Direct two-step difference: Hyperbolic Functions.* We consider Poisson integrals of functions g where $A_n = \frac{g^{(n)}(0)}{n!}$ satisfies a recurrence formula with two step difference. More precisely, when both t and s are complex numbers, it can be shown that the function

$$g_{t,s}(x) = \frac{\sqrt{t^2 - 4s} \cosh\left(\frac{\sqrt{t^2 - 4s}}{2}x\right) A_0 + \sinh\left(\frac{\sqrt{t^2 - 4s}}{2}x\right) (tA_0 + 2A_1)}{\sqrt{t^2 - 4s}} e^{-\frac{tx}{2}}, \quad (5.18)$$

is the only real analytic function at 0 with $A_n = \frac{g^{(n)}(0)}{n!}$ satisfying

$$(n+1)(n+2)A_{n+2} + t(n+1)A_{n+1} + sA_n = 0. \quad (5.19)$$

It is easy to see that

$$\int_0^\infty r^\gamma e^{-\alpha r} g_{t,s}(\beta r) dr = \frac{1}{\alpha^{\gamma+1}} u\left(\frac{\beta}{\alpha}\right) \quad (5.20)$$

where $u(x) = \sum_{n=0}^\infty B_n x^n$ and $B_n = A_n \Gamma(n + \gamma + 1)$. By (5.19), it follows that

$$(n+2)(n+1)B_{n+2} + t(n+1)(n+\gamma+2)B_{n+1} + \{sn^2 + (4\gamma+6)n + s(\gamma+1)(\gamma+2)\}B_n = 0.$$

Thus, by Theorem 1.1, it follows that u is the solution of the initial value problem

$$\begin{aligned} (\gamma+1)(\gamma+2)su + (t(2+\gamma) + (4+2\gamma)sx)u' + (1+tx+sx^2)u'' &= 0, \\ u(0) = \Gamma(\gamma+1)A_0, \quad u'(0) = \Gamma(\gamma+2)A_1. \end{aligned} \quad (5.21)$$

The function u satisfying (5.21) can be written explicitly in terms of the exponential and the inverse trigonometric functions. In particular, when $t=0$ and $s=1$, the initial value problem (5.21) has the solution

$$u(x) = \frac{A_1 \sin((1+\gamma) \arctan(x)) + A_0 \cos((1+\gamma) \arctan(x))}{(\sqrt{1+x^2})^{1+\gamma} (\Gamma(1+\gamma))^{-1}}. \quad (5.22)$$

Remark 5.1. When $t=0$ and $s=1$, the function $g_{0,1}$ is a linear combination of the trigonometric functions $\sin x$ and $\cos x$. Therefore, the integral (5.20) can be evaluated using (5.16) and (5.17). We thus have an elementary derivation of the not so obvious formulas tabulated in [3].

$${}_2F_1\left(\frac{\gamma+1}{2}, \frac{\gamma+2}{2}, \frac{1}{2}, -x^2\right) = \frac{\cos((1+\gamma) \arctan(x))}{(\sqrt{1+x^2})^{1+\gamma}}, \quad (5.23)$$

$${}_2F_1\left(\frac{\gamma+2}{2}, \frac{\gamma+3}{2}, \frac{3}{2}, -x^2\right) = \frac{\sin((1+\gamma) \arctan(x))}{x(1+\gamma)(\sqrt{1+x^2})^{1+\gamma}}, \quad (5.24)$$

for $\gamma > -1$ and real x .

5.3.2. *Indirect Two-step difference: Modified Bessel Function.* Our model example is the following integral tabulated in [3, p. 734] which concerns the modified Bessel function:

$$\int_0^\infty r e^{-\alpha r} K_0(\beta r) dr = \frac{1}{\alpha^2 - \beta^2} \left(\frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \ln \left[\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right] - 1 \right) \quad (5.25)$$

for $\alpha, \beta > 0$.

The derivation of (5.25) is simple. In fact, by making use of the power series representation of $e^{-\alpha r}$ and (4.3), we have that the left hand side of (5.25) is $\frac{1}{\beta^2} u(-\frac{\alpha}{\beta})$ where $u(x) = \sum_{n=0}^\infty A_n x^n$ with $A_n = 2^n [\Gamma(\frac{n+1}{2})]^2 / n!$. By noticing that $A_{n+2}/A_n = (n+2)/(n+1)$ and using Theorem 1.1, we obtain that u is the solution of the initial value problem

$$(x - x^3)u' - (2x^2 + 1)u + 1 = 0, \quad u(0) = \pi.$$

Hence, (5.25) follows.

6. MORE ON TWO STEP DIFFERENCE EQUATIONS

As pointed out in remark (iii) in the introduction, Theorem 1.1 is not restricted to evaluate integrals. In order to illustrate this, we shall obtain in this section generating functions corresponding to sequences (A_n) satisfying recurrence formulas with 2 consecutive terms which would allow us to derive new as well as old identities. The two steps difference equations that concern us in this section are those that lead to first order initial value problems. In fact, we consider the following type of difference equations

$$(\alpha_2 n + \beta_2)A_{n+2} + (\alpha_1 n + \beta_1)A_{n+1} + (\alpha_0 n + \beta_0)A_n + c = 0, n \geq 0 \quad (6.1)$$

for some constants $\alpha_i, \beta_i, 0 \leq i \leq m$ and c .

By Theorem 1.1, it follows that the generating function

$$u(x) = \sum_{n=0}^\infty A_n x^n$$

is the solution of the initial value problem

$$P(x)u'(x) + Q(x)u(x) + R(x) = 0, \quad u(0) = A_0 \quad (6.2)$$

where $P(x) = \alpha_2 x^3 + \alpha_1 x^2 + \alpha_0 x$, $Q(x) = \beta_0 x^2 + (\beta_1 - \alpha_1)x + (\beta_2 - 2\alpha_2)$, and $R(x) = (\beta_1 - \alpha_1)A_0 x + (\beta_2 - 2\alpha_2)A_0 + (\beta_2 - \alpha_2)A_1 x + \frac{cx^3}{1-x}$.

6.1. Some Interesting Generating Functions. Our first example of generating functions that obey IVP of the form (6.2) come from our every day calculus trigonometric integrals. Anyone who has taught a first course of calculus has come across some trigonometric integrals for which the students have to come up with the right recurrence formula. For these type of integrals and for $0 < b < \frac{\pi}{2}$ and $0 < a \leq \pi$, we have generated the following table:

Sequence	Generating function
$\int_0^b \tan^n t dt$	$\frac{b-x \ln(\cos b-x \sin b)}{1+x^2}$
$\int_0^b \tan^{2n} t dt$	$\frac{\ln(2 \cos^2 b) - \ln(1-x+(1+x) \cos 2b) + \tan^2 b}{2(1+x)}$
$\int_0^b \tan^{2n+1} t dt$	$\frac{b-x \ln(\cos b-x \sin b)}{1+x^2} - \frac{\ln(2 \cos^2 b) - \ln(1-x+(1+x) \cos 2b) + \tan^2 b}{2(1+x)}$
$\int_0^a \cos^n t dt$	$\frac{\arctan\left(\frac{\sqrt{1-x^2} \sin a}{x-\cos a}\right)}{\sqrt{1-x^2}}$
$\int_0^a \sin^n t dt$	$\frac{\arcsin x - \arctan\left(\frac{x-\sin a}{\sqrt{1-x^2} \cos a}\right)}{\sqrt{1-x^2}}$

TABLE 1. Generating functions of calculus trigonometric integrals

Our next example in this section is the generating function of the famous Schröder numbers $s(n)$ believed to be known to Hipparchus 190–127 B. C. [2]. The numbers $s(n)$ are given by the recurrence formula [14]:

$$3(2n - 1)s(n) = (n + 1)s(n + 1) + (n - 2)s(n - 1), \quad n \geq 2,$$

$$s(1) = s(2) = 1.$$

By Setting $A_n = s(n + 1)$, it can be shown that A_n satisfies (6.1) with $\alpha_0 = -1, \alpha_1 = 6, \beta_1 = 9, \alpha_2 = -1, \beta_2 = -3$, and $\alpha_3 = \beta_3 = c = 0$. Thus by (6.2), we obtain that the function $u(x) = \sum_{n=0}^{\infty} A_n x^n$ is the solution of the initial value problem

$$(6x^2 - x - x^3)u'(x) + (3x - 1)u(x) - x + 1 = 0, \quad u(0) = 1$$

which immediately implies the following formula.

$$\sum_{n=1}^{\infty} s(n)x^n = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4}. \tag{6.3}$$

Hence,

$$s(n) = \frac{1}{n!} \frac{d^n}{dx^n} \left(\frac{1 + x - \sqrt{1 - 6x + x^2}}{4} \right) \Big|_{x=0}.$$

For further results about Schröder numbers, one can consult [13].

6.2. Two Infinite Sums. Evaluating infinite sums is not always an easy exercise. Especially, when the terms of the series are expressed in terms of very involved functions like the ${}_2F_1$ function and many other special functions. However, in the following two theorems, we illustrate the effectiveness of Theorem 1.1 in evaluating such kind of sums.

Theorem 6.1. For $-1 \leq x < 1$ and $\alpha > 0$, we have

$$\begin{aligned} & (1-x) \sum_{n=0}^{\infty} \frac{{}_2F_1(n+\alpha, n+\alpha; n+\alpha+1; -1)}{n+\alpha} x^n \\ &= \frac{1}{\alpha} {}_2F_1(\alpha, \alpha; \alpha+1; -1) - \frac{x^2 {}_2F_1(\alpha+1, 1; \alpha+2; \frac{x}{2})}{(\alpha+1)2^{\alpha+1}} - \frac{x}{\alpha 2^\alpha} \end{aligned} \quad (6.4)$$

and

$$\begin{aligned} & \frac{(1-x)}{\alpha} F_1(\alpha; 1; \alpha-1; \alpha+1; x-1, -1) \\ &= \frac{1}{\alpha} {}_2F_1(\alpha, \alpha; \alpha+1; -1) - \frac{x^2 {}_2F_1(\alpha+1, 1; \alpha+2; \frac{x}{2})}{(\alpha+1)2^{\alpha+1}} - \frac{x}{\alpha 2^\alpha}. \end{aligned} \quad (6.5)$$

Proof. To prove (6.4), we first notice that

$$a_n = \frac{{}_2F_1(n+\alpha, n+\alpha; n+\alpha+1; -1)}{n+\alpha} = \int_0^1 (1+t)^{-n-\alpha} t^{n+\alpha-1} dt.$$

Thus, an integration by parts implies that

$$a_{n+1} - a_n = -\frac{2^{-n-\alpha}}{n+\alpha}$$

which by (6.1) and (6.2) implies that the function

$$u(x) = \sum_{n=0}^{\infty} \frac{{}_2F_1(n+\alpha, n+\alpha; n+\alpha+1; -1)}{n+\alpha} x^n$$

is the solution of the initial value problem (6.2) with initial condition

$$a_0 = \frac{{}_2F_1(\alpha, \alpha; \alpha+1; -1)}{\alpha}$$

and $P(x) = x^3 - 3x^2 + 2x$, $Q(x) = \alpha x^2 - (3\alpha - 1)x + 2(\alpha - 1)$, and $R(x) = ((2\alpha - 1)a_0 + 2^{-\alpha})x - 2(\alpha - 1)a_0$.

By solving the obtained differential equation and making use of the initial condition and the fact that the aimed solution u is real analytic at 0, we immediately obtain (6.4).

Next, to obtain (6.5), notice that

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} \frac{{}_2F_1(n+\alpha, n+\alpha; n+\alpha+1; -1)}{n+\alpha} x^n \\ &= \int_0^1 \sum_{n=0}^{\infty} (1+t)^{-n-\alpha} t^{n+\alpha-1} x^n dt = \int_0^1 \left(\sum_{n=0}^{\infty} \left(\frac{xt}{1+t} \right)^n \right) t^{-1} \left(\frac{t}{1+t} \right)^{\alpha} dt \\ &= \int_0^1 \frac{t^{\alpha-1}}{1-(x-1)t} \frac{1}{(1+t)^{\alpha-1}} dt. \end{aligned}$$

Thus, by (2.7), we get

$$u(x) = \frac{1}{\alpha} F_1(\alpha; 1, \alpha-1; \alpha+1; x-1, -1). \quad (6.6)$$

Then, by (6.6) and (6.4), we obtain (6.5). This completes the proof. \square

Our final result in this section concerns the upper incomplete gamma function $\Gamma(a, x)$. For $\alpha > 0$, let

$$S(\alpha, t, x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n, t)}{\Gamma(n+\alpha)} x^n. \quad (6.7)$$

Then, we have the following formula.

Theorem 6.2. *Let $\alpha > 0$ and t be real. Then for $x \in [-1, 1]$, we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n, t)}{\Gamma(\alpha+n)} x^n \\ &= \frac{x^{1-\alpha} e^{-t(1-x)} \{\Gamma(\alpha) - (\alpha-1)\Gamma(\alpha-1, tx)\} - t^{\alpha-1} e^{-t} + \Gamma(\alpha, t)}{(1-x)\Gamma(\alpha)} \end{aligned} \quad (6.8)$$

and

$$\sum_{n=0}^m \frac{\Gamma(\alpha+n, t)}{\Gamma(\alpha+n)} x^n = S(\alpha, t, x) - x^{m+1} S(\alpha+m, t, x) \quad (6.9)$$

where $S(\alpha, t, x)$ is the right hand side of (6.8).

Proof. First, by the definition of the upper incomplete gamma function and integration by parts, it follows that $A_n = \Gamma(\alpha+n, t)/\Gamma(n+\alpha)$ satisfies the difference equation

$$(n+\alpha+1)A_{n+2} - (n+\alpha+t+1)A_{n+1} + tA_n = 0.$$

Therefore, $S(\alpha, t, x)$ is the solution of the first order initial value problem (6.2) with

$$\begin{aligned} P(x) &= x - x^2, & Q(x) &= \alpha - 1 - (\alpha+t)x + tx^2 \\ R(x) &= (\alpha-1)A_0 + (tA_0 + \alpha A_0 + A_1 - \alpha A_1)x. \end{aligned}$$

Hence, the result follows. The proof of (6.9) follows a similar argument as the corresponding one for (6.5). \square

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