EXTENDING LANDAU'S THEOREM ON DIRICHLET SERIES WITH NON-NEGATIVE COEFFICIENTS

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Abstract. A classical theorem of Landau states that, if an ordinary Dirichlet series has non-negative coefficients, then it has a singularity on the real line at its abscissa of convergence. In this article, we relax the condition on the coefficients while still arriving at the same conclusion. Specifically, we write a_n as $|a_n|e^{i\theta_n}$ and we consider the sequences $\{|a_n|\}$ and $\{\cos\theta_n\}$. Let $M \in \mathbb{N}$ be given. The condition on $\{|a_n|\}$ is that, dividing the sequence sequentially into vectors of length M, each vector lies in a certain convex cone $B \subset [0,\infty)^M$. The condition on $\{\cos\theta_n\}$ is (roughly) that, again dividing the sequence sequentially into vectors of length M, each vector lies in the negative of the polar cone of B. We demonstrate the additional freedom allowed in choosing the θ_n, compared to Landau's Theorem. We also obtain sharpness results.

1. Introduction

A (ordinary) Dirichlet series is a function of the following form, with $a_n \in \mathbb{C}$:

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad s \in \mathbb{C}. \tag{1}$$

For $s = \sigma + it \in \mathbb{C}$, we denote the real part of s by $\Re s$. The standard region on which a Dirichlet series might be expected to converge is a right half plane, we denote these by

$$\Omega_\sigma = \{s \in \mathbb{C} : \Re s > \sigma\}$$

and its closure will be written $\overline{\Omega}_\sigma$. A Dirichlet series has several different "regions of convergence" Ω_σ, with several different abscissa σ accordingly. The abscissa most often considered are:

$$\begin{aligned}
\sigma_a &= \inf\{\sigma : \sum a_n n^{-s} \text{ converges absolutely for } s \in \Omega_\sigma\},\\
\sigma_u &= \inf\{\sigma : \sum a_n n^{-s} \text{ converges uniformly on } \Omega_\sigma\},\\
\sigma_b &= \inf\{\sigma : \sum a_n n^{-s} \text{ converges to a bounded function on } \Omega_\sigma\},\\
\sigma_c &= \inf\{\sigma : \sum a_n n^{-s} \text{ converges for all } s \in \Omega_\sigma\}.
\end{aligned}$$

It is a basic result that the function f defined by (1) is holomorphic on the open region Ω_{σ_c}. Further relations among these abscissa, the coefficients $\{a_n\}$, and the function f are of considerable interest. Some of the standard results are the following ones.

- $\sigma_a - \sigma_c \leq 1$ (a basic result), and this is sharp (ex. the alternating zeta function $\sum(-1)^{n+1}n^{-s}$).
- $\sigma_u = \sigma_b$ [6], henceforth we will denote this abscissa by σ_b.
- $\sigma_a - \sigma_b \leq 1/2$ [5], and this is sharp [4].

For other standard results in analytic number theory and Dirichlet series, we refer the interested reader to [1].

There has been recent interest in applying tools from modern analysis to Dirichlet series (see the survey of Hedenmalm [8]). A short list (non-exhaustive in both topics and articles within those topics) includes the interpolation problem within Hilbert spaces of Dirichlet series [13], the multiplier algebras of Hilbert spaces of Dirichlet series [9, 12], Carleson-type theorems for Dirichlet series [10, 3], and composition operators on spaces of Dirichlet series [2].

We mention the above results for contrast, because our result will be "classic" in both statement and proof, and we will investigate Dirichlet series which (among other things) satisfy

$$\sigma_a = \sigma_c. \tag{2}$$

Specifically, we are interested in extending the following theorem of Landau.

Theorem 1. [11, p. 697–699] *Suppose that $f(s) = \sum a_n n^{-s}$ has abscissa of convergence equal to 0. If $a_n \in \mathbb{R}, a_n \geq 0$ for all n then f does not extend holomorphically to a neighborhood of $s = 0$.*

Logically, the property that must account for the situation $\sigma_c < \sigma_a$ is cancellation among the coefficients $\{a_n\}$. Therefore, once we strictly limit cancellation among the $\{a_n\}$, (2) should follow. A straightforward way to do this is to require $a_n \geq 0$, and the above theorem confirms this (note that the absence of a holomorphic extension about $s = 0$ is stronger than (2)).

It is a natural question to ask whether we could impose less strict conditions on the $\{a_n\}$ and still arrive at the same conclusion. A paper of Fekete from 1910 states the following result.

Theorem 2. [7] *Suppose that $f(s) = \sum a_n n^{-s}$ has abscissa of convergence equal to 0. Write $a_n = |a_n|e^{i\theta_n}$. If there is some $\gamma > 0$ such that $\cos\theta_n \geq \gamma$ for all n, then f does not extend holomorphically to a neighborhood of $s = 0$.*

This result generalizes Theorem 1. Another result, by Szász in 1922, gives the following theorem.

Theorem 3. [14] *Let $F_1(s)$ and $F_2(s)$ be the Dirichlet series whose coefficients are the real part, and imaginary part (respectively), of the coefficients a_n. If either $F_1(s)$ or $F_2(s)$ has a singularity on the real line at $\sigma_c(f)$, then f has a singularity there as well.*

Note that the result of Szász, together with Landau's Theorem, implies the result of Fekete. This is because, if $\cos\theta_n \geq \gamma$, then $\Re a_n$ is non-negative and therefore (by Landau's Theorem) F_1 has a singularity at the point $\sigma_c(F_1)$. Furthermore, the condition on θ_n implies $\gamma|a_n| \leq \Re a_n$, which in turn yields the middle inequality in this chain:

$$\sigma_c(f) \leq \sigma_a(f) \leq \sigma_c(F_1) \leq \sigma_c(f).$$

We see that F_1 has a singularity at $\sigma_c(f)$ and then by Szász's theorem, f has a singularity there.

Our result is along the same lines as these three theorems; we develop a condition on the coefficients which will imply that f has a singularity on the real line at its abscissa of absolute convergence. Our result generalizes that of Fekete, although not the result of Szász.

In our result, the conditions on the $\{a_n\}$ are expressed as certain restrictions on the sequence $|a_n|$, and related restrictions on the sequence $\{\cos(\theta_n)\}$. We will see that as the restrictions on $|a_n|$ are relaxed, the restrictions on $\cos(\theta_n)$ become more strict. The theorem of Fekete falls on one end of this spectrum, with no requirements on $|a_n|$ and strict requirements on $\cos(\theta_n)$.

Let us fix $M \in \mathbb{N}$. For $\rho \in (0, \infty)$, let us define the cone $B^\rho \subset [0,\infty)^M$ by

$$B^\rho = \left\{\beta = (\beta_1, \ldots, \beta_M) \in [0,\infty)^M \ : \beta_1 \leq \rho\beta_2 \leq \rho^2\beta_3 \leq \cdots \leq \rho^{M-1}\beta_M\right\}. \tag{3}$$

Note the inclusion

$$\rho_1 < \rho_2 \implies B^{\rho_1} \subset B^{\rho_2}.$$

The standard inner product in Euclidean space will be denoted $x \cdot y$. We will use the following notation for the negative of the polar cone of a convex cone $C \subset \mathbb{R}^M$:

$$C_\sharp = \{x \in \mathbb{R}^M : x \cdot c \geq 0 \text{ for all } c \in C\}.$$

Note that, if $M = 1$, then $B^\rho = B^\rho_\sharp = [0, \infty)$ for any value of ρ.

Our main theorem is the following result.

Theorem 4. *Suppose that $f(s) = \sum a_n n^{-s}$ has abscissa of absolute convergence equal to 0. Write $a_n = |a_n|e^{i\theta_n}$, and fix $M \in \mathbb{N}$. Suppose that there exists $\rho > 0$ and $\gamma > 0$ such that, for all $l = 0, 1, \ldots$, we have*

$$(|a_{Ml+1}|, |a_{Ml+2}|, \ldots, |a_{Ml+M}|) \in B^\rho \tag{4}$$

$$(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \ldots, \cos(\theta_{Ml+M})) \in B^\rho_\sharp + \gamma(1, 1, \ldots, 1). \tag{5}$$

Then f does not have a holomorphic extension to a neighborhood of $s = 0$.

First, note that condition (4) is not a "global" growth or decay condition; with $M = 2$ it is satisfied by $\rho, 1, \rho, 1, \ldots$. Second, noting that

$$C^1 \subset C^2 \implies C^2_\sharp \subset C^1_\sharp$$

we see that if ρ increases then B^ρ becomes larger and therefore $B^\rho_\sharp$ becomes smaller. In this sense, (4) and (5) are "dual" to one another; the amount of restriction on $|a_n|$ is inversely proportional to the restriction on $\cos(\theta_n)$. Third, note that the purpose of requiring $\gamma > 0$ is to keep the cosine vector inside the cone $B^\rho_\sharp$ and bounded away from its boundary, the specific value of γ does not otherwise affect the rest of the theorem (for further explanation, see the sharpness result below). Fourth, note that if we take $M = 1$, we obtain the equivalent version of the result of Fekete.

We discuss for a moment the hypothesis $\sigma_c = 0$ from Theorems 1 and 2, which appears to differ from the hypothesis $\sigma_a = 0$ in our main Theorem. The condition $\cos\theta_n \geq \gamma$ from Theorem 2 implies $\Re a_n \geq \gamma|a_n|$ and therefore

$$\Re \sum a_n n^{-\sigma} \geq \gamma \sum |a_n| n^{-\sigma}$$

from which it follows that $\sigma_c = \sigma_a$. So, although Theorems 1 and 2 are stated with the hypothesis $\sigma_c = 0$, in fact the hypothesis on the coefficients a_n implies that $\sigma_c = \sigma_a$ and so each theorem could be equivalently stated with the hypothesis $\sigma_a = 0$. Since we always have $\sigma_c \leq \sigma_a$, stating the theorems with the hypothesis "$\sigma_a = 0$" would be arguably more useful for the reader who could use the theorem to immediately conclude $\sigma_c = \sigma_a$ without further analysis. Our main result has the same property that the hypothesis on the coefficients, namely equations (4) and (5), in fact implies $\sigma_a = \sigma_c$. We have stated our main theorem with the hypothesis $\sigma_a = 0$, but we leave Theorems 1 and 2 in their historical form.

In Theorem 2, we saw that with no restrictions on the $|a_n|$ we are free to choose θ_n with $\cos(\theta_n) \in [\gamma, 1]$; applying this to a group of M terms, we are free to choose

$$(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \ldots, \cos(\theta_{Ml+M})) \in [\gamma, 1]^M.$$

The M-dimensional volume of the set of admissable values of cosines is less than or equal to $1^M = 1$, for Theorem 2. In Theorem 4, we have placed restrictions on the sequence $|a_n|$. Therefore, Theorem 4 is only interesting if we can considerably increase the freedom in choosing θ_n, beyond the amount in Theorem 2. As mentioned above, the case $M = 1$ is equivalent to Theorem 2, so we would like to show that we have additional freedom in choosing θ_n when $M \geq 2$. This is demonstrated in Section 4, where we show that the volume of the set of admissable values of cosines is greater than 1 for all $M \geq 2$ and all ρ (the interested reader can further quantify this result). Therefore, Theorem 4 provides more "freedom" in choosing the θ_n, compared to Theorem 2.

We also obtain a sharpness result, in two parts. First, we cannot reduce γ to zero. Second, if $M \geq 2$, the cone giving the possible values for $\cos\theta_n$ cannot be made any "wider," meaning that one cannot replace $B_\sharp^\rho$ with $B_\sharp^{\rho'}$ for $\rho' < \rho$ (there is no such statement for $M = 1$ because every cone $B_\sharp^\rho$ equals $[0, \infty)$ when $M = 1$). Here, "∂X" denotes the boundary of the set X.

Proposition 5. *(I) ($\gamma > 0$ is sharp): Let $M \in \mathbb{N}, \rho \in (0, \infty)$ be fixed. For any*
$\delta \in (\partial B_\sharp^\rho) \cap [-1, 1]^M$, there exists $\{a_n\}$ such that

- *$\sum a_n n^{-s}$ has $\sigma_a = 0$.*
- *$(|a_{Ml+1}|, |a_{Ml+2}|, \ldots, |a_{Ml+M}|) \in B^\rho$.*
- *$(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \ldots, \cos(\theta_{Ml+M})) = \delta$ [This is (5) with $\gamma = 0$].*
- *$\sum a_n n^{-s}$ has a holomorphic extension to a neighborhood of $s = 0$.*

(II) ($B^\rho, B_\sharp^\rho$ is sharp): For any $M \geq 2$ and any $0 < \rho' < \rho$ there exists $\{a_n\}$ and $\gamma > 0$ such that

- *$\sum a_n n^{-s}$ has $\sigma_a = 0$.*
- *$(|a_{Ml+1}|, |a_{Ml+2}|, \ldots, |a_{Ml+M}|) \in B^\rho$.*
- *$(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \ldots, \cos(\theta_{Ml+M})) \in B_\sharp^{\rho'} + \gamma(1, 1, \ldots, 1)$.*
- *$\sum a_n n^{-s}$ has a holomorphic extension to a neighborhood of $s = 0$.*

In Section 2, we examine the proof of Landau's Theorem. In Section 3, we prove Theorem 4. In Section 4, we estimate the amount of "freedom" in choosing θ_n, and in Section 5 we prove the sharpness result in Proposition 5.

2. Examining the Proof of Landau's Theorem

Our result will build on a standard proof of Landau's Theorem, so we begin by reviewing this proof. There is a key step in the proof which will become the starting point of our main result.

Proof of Theorem 1. We begin by supposing *only* that $f(s) = \sum a_n n^{-s}$ has abscissa of absolute convergence equal to 0. The condition $a_n \geq 0$ is not yet assumed; when it is used, we will indicate this explicitly.

For contradiction, we assume that f does extend holomorphically to a neighborhood of 0; suppose that f is holomorphic on $\mathbb{D}(0, 3\epsilon)$, for some $\epsilon > 0$. We have

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} a_n n^{-\epsilon} n^{-(s-\epsilon)} \\ &= \sum_{n=1}^{\infty} a_n n^{-\epsilon} \exp(-(s-\epsilon)\log n) \\ &= \sum_{n=1}^{\infty} a_n n^{-\epsilon} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k (\log n)^k (s-\epsilon)^k}{k!} \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{\infty} a_n n^{-\epsilon} \frac{(-1)^k (\log n)^k (s-\epsilon)^k}{k!} \right\}. \end{aligned}$$

This double series converges absolutely for $|s - \epsilon| < \epsilon$, since the sum of the absolute values can be re-arranged to equal

$$\sum_{n=1}^{\infty} |a_n| n^{-(\epsilon - |s-\epsilon|)}$$

which is finite by assumption. Therefore, we rearrange the double series to obtain

$$f(s) = \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{k!} \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \right\} (s-\epsilon)^k.$$

We see that this is the power series for f about the point $s = \epsilon$. We have only asserted the convergence of this power series for $|s - \epsilon| < \epsilon$. However, by the assumption that f is holomorphic on $\mathbb{D}(0, 3\epsilon)$, it must be the case that this power series in fact converges absolutely for $|s - \epsilon| < 3\epsilon$, since $\mathbb{D}(\epsilon, 3\epsilon) \subset \big(\mathbb{D}(0, 3\epsilon) \cup \{s : \Re s > 0\} \big)$. Therefore, in particular we have finiteness of the expression

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left| \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \right| (2\epsilon)^k. \tag{6}$$

We could obtain a contradiction *if* we could obtain finiteness of the expression

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \sum_{n=1}^{\infty} |a_n| n^{-\epsilon} (\log n)^k \right\} (2\epsilon)^k. \tag{7}$$

This would create a contradiction because, if (7) were finite, then we could rearrange (7) to obtain

$$\sum_{n=1}^{\infty} |a_n| n^{-(-\epsilon)} < \infty.$$

This would mean that $\sum a_n n^{-s}$ converges absolutely at $s = -\epsilon$ (for example), a contradiction.

To complete the standard proof, let us now assume $a_n \geq 0$. With this requirement on the a_n, we note that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left| \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \right| (2\epsilon)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \sum_{n=1}^{\infty} |a_n| n^{-\epsilon} (\log n)^k \right\} (2\epsilon)^k.$$

Therefore, we obtain finiteness of (7) and the proof is complete.

□

Examining this proof, we see that the condition $a_n \geq 0$ is only used to prove the finiteness of (7), given the finiteness of (6). Any other condition which establishes this implication will also suffice to prove the theorem. We will focus on a quite straightforward way to establish this implication, namely if we have that the expression in (7) is less than or equal to a constant C times the expression in (6) for all sufficiently small ϵ. In other words, if there are constants C and ϵ_0 such that

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \sum_{n=1}^{\infty} |a_n| n^{-\epsilon} (\log n)^k \right\} (2\epsilon)^k \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{k!} \left| \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \right| (2\epsilon)^k \text{ for all } \epsilon < \epsilon_0. \end{aligned}$$

We will use an even simpler sufficient condition which implies the above inequality. We will require a stronger statement where the absolute value is replaced by the real part, and we will require the inequality to hold term-by-term with a common constant. In other words, we require that there are constants C and ϵ_0 such that

$$\sum_{n=1}^{\infty} |a_n| n^{-\epsilon} (\log n)^k \leq C \Re \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \text{ for all } k \geq 0, \text{ for all } \epsilon < \epsilon_0.$$

To summarize, we have the following theorem.

Theorem 6 (Landau and Fekete Theorems, Re-formulated)**.** *Suppose that* $f(s) = \sum a_n n^{-s}$ *has abscissa of absolute convergence equal to* 0*. If there are constants* C *and* ϵ_0 *such that*

$$\sum_{n=1}^{\infty} |a_n| n^{-\epsilon} (\log n)^k \leq C \, \Re \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \qquad \text{for all } k \geq 0 \text{ and for all } \epsilon < \epsilon_0 \tag{8}$$

then f *does not have a holomorphic extension to a neighborhood of* 0*.*

We can see how both Landau's and Fekete's Theorems satisfy equation (8): Landau's Theorem assumes $|a_n| = \Re a_n$, and Fekete's Theorem assumes $|a_n| \leq \gamma^{-1} \Re a_n$.

3. Proof of the Main Theorem

In this section, we prove Theorem 4.

Our strategy is to show that (8) holds. Fix $M \in \mathbb{N}$ and write

$$\sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k = \sum_{l=0}^{\infty} \sum_{j=1}^{M} a_{Ml+j} (Ml + j)^{-\epsilon} (\log(Ml + j))^k$$

which yields

$$\begin{aligned} &\Re \sum_{n=1}^{\infty} a_n n^{-\epsilon} (\log n)^k \\ &= \sum_{l=0}^{\infty} \sum_{j=1}^{M} \Re a_{Ml+j} (Ml + j)^{-\epsilon} (\log(Ml + j)\,)^k \\ &= \sum_{l=0}^{\infty} \sum_{j=1}^{M} |a_{Ml+j}| \cos(\theta_{Ml+j}) (Ml + j)^{-\epsilon} (\log(Ml + j)\,)^k . \end{aligned}$$

We wish to show that conditions (4) and (5) on the group of coefficients $a_{Ml+1}, \ldots, a_{Ml+M}$ will imply the existence of some $c > 0$ (independent of k, l, ϵ) and some ϵ_0 such that

$$\begin{aligned} &\sum_{j=1}^{M} |a_{Ml+j}| \cos(\theta_{Ml+j}) (Ml + j)^{-\epsilon} (\log(Ml + j)\,)^k \\ &\quad \geq c \Big(\sum_{j=1}^{M} |a_{Ml+j}| (Ml + j)^{-\epsilon} (\log(Ml + j)\,)^k \Big) \qquad \text{for all } \epsilon < \epsilon_0 \, . \end{aligned} \tag{9}$$

Once (9) holds, we have

$$\Re\sum_{n=1}^{\infty} a_n n^{-\epsilon}(\log n)^k \geq c\sum_{n=1}^{\infty} |a_n| n^{-\epsilon}(\log n)^k \text{ for all } \epsilon < \epsilon_0$$

which shows that (8) holds, and the proof of Theorem 4 is complete.

The next idea is that

$$(Ml+j)^{-\epsilon} = (Ml)^{-\epsilon} + \text{small}$$

and so we can obtain (9) from a simpler condition that does not involve $(Ml+j)^{-\epsilon}$, by applying the Taylor expansion of $(Ml+j)^{-\epsilon}$ to each side of (9). (The simpler condition is (11)).

We begin with the right hand side of (9). By Taylor's expansion, we write

$$(Ml+j)^{-\epsilon} = (Ml)^{-\epsilon} + A, \qquad |A| \leq \epsilon(Ml)^{-\epsilon} l^{-1}$$

and therefore we have

$$\begin{aligned}&\sum_{j=1}^{M} |a_{Ml+j}|(Ml+j)^{-\epsilon}(\log(Ml+j))^k \\ &\qquad \leq (Ml)^{-\epsilon}\big(1+\epsilon l^{-1}\big)\sum_{j=1}^{M} |a_{Ml+j}|(\log(Ml+j))^k. \qquad (10)\end{aligned}$$

Suppose that the following inequality held for γ independent of l, k:

$$\sum_{j=1}^{M} |a_{Ml+j}|(\log(Ml+j))^k \leq \gamma^{-1}\sum_{j=1}^{M} |a_{Ml+j}|\cos(\theta_{Ml+j})(\log(Ml+j))^k. \qquad (11)$$

Applying the Taylor expansion to the left hand side in (9) (estimating $\cos(\theta) \leq 1$), we define

$$\tilde{A} = \epsilon(Ml)^{-\epsilon} l^{-1}\sum_{j=1}^{M} |a_{Ml+j}|(\log(Ml+j))^k$$

and we have

$$\begin{aligned}
&\sum_{j=1}^{M} |a_{Ml+j}| \cos(\theta_{Ml+j})(Ml+j)^{-\epsilon}(\log(Ml+j))^k \\
&\geq (Ml)^{-\epsilon} \sum_{j=1}^{M} |a_{Ml+j}| \cos(\theta_{Ml+j})(\log(Ml+j))^k - \tilde{A} \\
&\geq (Ml)^{-\epsilon} \gamma \sum_{j=1}^{M} |a_{Ml+j}|(\log(Ml+j))^k - \tilde{A} \qquad \text{[by (11)]} \\
&= (Ml)^{-\epsilon} \left(\gamma - \epsilon l^{-1}\right) \sum_{j=1}^{M} |a_{Ml+j}|(\log(Ml+j))^k \\
&\geq \left(\gamma - \epsilon l^{-1}\right) \left(1 + \epsilon l^{-1}\right)^{-1} \sum_{j=1}^{M} |a_{Ml+j}|(Ml+j)^{-\epsilon}(\log(Ml+j))^k \quad \text{[by (10)]}.
\end{aligned}$$

With $\epsilon < 1$ we have $\left(\gamma - \epsilon l^{-1}\right)\left(1 + \epsilon l^{-1}\right)^{-1} \geq \left(\gamma - l^{-1}\right)\left(1 + l^{-1}\right)^{-1}$. We may assume that $a_n = 0$ for all small n (since $\sum_{n=1}^{\infty} a_n n^{-s}$ has a holomorphic extension if and only if $\sum_{n=N}^{\infty} a_n n^{-s}$ does), and therefore we may assume that we are concerned only with large l. For l large (depending only on γ) we have $\left(\gamma - l^{-1}\right)\left(1 + l^{-1}\right)^{-1} \geq \gamma/2$, and therefore (9) holds (with $c = \gamma/2$) for all sufficiently small ϵ, independent of k, l, as long as (11) holds.

Therefore, we only need to show that conditions (4) and (5) imply inequality (11) and then the proof is complete.

First, in order to make use of condition (5), we will use a more direct description of $B^{\rho}_{\sharp}$, by writing B^{ρ} as the convex cone generated by a finite point set.

Proposition 7. *Let $x^{(r)} \in \mathbb{R}^M, r = 1, \ldots, M$ be defined by*

$$x^{(r)} = (0, \ldots, 0, \rho^{-r}, \rho^{-(r+1)}, \ldots, \rho^{-M}).$$

Then B^{ρ} equals the positive linear span of the $\{x^{(r)}\}$.

Corollary 8.

$$B^{\rho}_{\sharp} = \left\{ y = (y_1, \ldots, y_M) : \sum_{j=r}^{M} \rho^{-j} y_j \geq 0 \text{ for all } r = 1, \ldots, M \right\}. \tag{12}$$

Proof of Proposition. We see that $x^{(r)} \in B^{\rho}$ is clear. If $\beta \in B^{\rho}$ then

$$\beta = \rho^1 \beta_1 x^{(1)} + \rho^2(\beta_2 - \rho^{-1}\beta_1)x^{(2)} + \cdots + \rho^M(\beta_M - \rho^{-1}\beta_{M-1})x^{(M)}.$$

Each coefficient is positive, so we have written B^{ρ} as a positive linear combination of the $x^{(r)}$. $\square$

To prove (11) we will use a summation-by-parts argument, in the following form.

Proposition 9 (A Version of Summation by Parts). *Suppose* $a_1, \ldots, a_M$ *and* $b_1, \ldots, b_M$ *are given. Define* $R_N = \sum_{j=N}^{M} b_j$. *Then*

$$\sum_{j=1}^{M} a_i b_i = a_1 R_1 + \sum_{j=2}^{M} (a_j - a_{j-1}) R_j.$$

Proof. The proof is a standard one. $\square$

Theorem 4 assumes the existence of particular values of ρ and γ; these are used below. Let us define

$$d_j = |a_{Ml+j}| (\log(Ml + j))^k \rho^j,$$

$$c_j = \cos(\theta_{Ml+j}) \rho^{-j}, \qquad C_j = \sum_{r=j}^{M} c_r,$$

$$g_j = \gamma \rho^{-j}, \qquad G_j = \sum_{r=j}^{M} g_r.$$

Condition (4) means that the sequence $|a_{Ml+j}|\rho^j$ is nondecreasing, and therefore d_j is nondecreasing; this is how we use condition (4). Condition (5) states that

$$(\cos(\theta_{Ml+1}) - \gamma, \ldots, \cos(\theta_{Ml+M}) - \gamma) \in B^{\rho}_{\sharp}.$$

By (12), this means that

$$\sum_{j=r}^{M} \rho^{-j} (\cos(\theta_{Ml+j}) - \gamma) \geq 0 \text{ for all } r = 1, \ldots, M$$

which we can write as $C_j \geq G_j$ for all $r = 1, \dots, M$; this is how we use condition (5). We can now prove (11).

$$\begin{aligned}
\sum_{j=1}^{M} |a_{Ml+j}| \cos(\theta_{Ml+j})(\log(Ml+j))^k &= \sum_{j=1}^{M} d_j c_j \\
&= d_1 C_1 + \sum_{j=2}^{M} (d_j - d_{j-1}) C_j \\
&\geq d_1 G_1 + \sum_{j=2}^{M} (d_j - d_{j-1}) G_j \qquad \{**\} \\
&= \sum_{j=1}^{M} d_j g_j \\
&= \gamma \sum_{j=1}^{M} |a_{Ml+j}| (\log(Ml+j))^k.
\end{aligned}$$

In line $\{**\}$, we use that d_j is nondecreasing, and that $C_j \geq G_j$.

This shows that conditions (4) and (5) imply the inequality (11), and so the proof of Theorem 4 is complete.

4. Volume Calculation

As we mentioned, Theorem 4 is only interesting if the restrictions on θ_n are broad enough to be a measurable improvement over the requirement $\cos\theta_n \geq \gamma$, which is the requirement in Theorem 2. As mentioned, taking $M = 1$ in Theorem 4 yields an equivalent version of Theorem 2, so in this section we only consider $M \geq 2$. In Theorem 4, we require

$$(\cos(\theta_{Ml+1}), \dots, \cos(\theta_{Ml+M})) \in B_\sharp^\rho + \gamma \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}.$$

We want to answer the question: "How much freedom do we have in choosing $\cos(\theta_{Ml+1}), \dots, \cos(\theta_{Ml+M})$?" One way to answer this is to measure the volume of the set of possible values for the vector $(\cos(\theta_{Ml+1}), \dots, \cos(\theta_{Ml+M}))$, i.e.,

$$Vol^{\mathbb{R}^M} \left[\left(B_\sharp^\rho + \gamma \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \right) \cap [-1,1]^M \right].$$

Since this is continuous in γ, we will estimate

$$Vol^{\mathbb{R}^M} \left[B_\sharp^\rho \cap [-1,1]^M \right]. \tag{13}$$

We will compare this volume to the value 1, since the requirement $\cos\theta_n \geq \gamma$ means

$$(\cos(\theta_{Ml+1}), \ldots, \cos(\theta_{Ml+M})) \in [\gamma, 1]^M$$

and $Vol^{\mathbb{R}^M}\left[\,[\gamma,1]^M\right] < 1$. Here, we show that the volume is nondecreasing in M, and then we carry out the estimate for $M = 2$; the interested reader can extend the result.

For this particular discussion, we will specify the dimension of the cone B^ρ by writing it as $B^{\rho,M}$. Examining equation (12) for dimension $M+1$ we see that, if $x_1 \geq 0$ is fixed, the constraint for $r = 1$ is superfluous because if $\sum_{j=2}^{M+1} \rho^{-j} x_j \geq 0$ then $\sum_{j=1}^{M+1} \rho^{-j} x_j \geq 0$ is also satisfied. However, the remaining constraints specify that the point defined by the last M coordinates be a member of $B_\sharp^{\rho,M}$. In other words,

$$B_\sharp^{\rho,M+1} \cap \left(\mathbb{R}^+ \times \mathbb{R}^M\right) = \mathbb{R}^+ \times B_\sharp^{\rho,M}.$$

Using this, one can show

$$B_\sharp^{\rho,M+1} \cap \left([0,1] \times [-1,1]^M\right) = [0,1] \times \left(B_\sharp^{\rho,M} \cap [-1,1]^M\right).$$

The set on the left hand side of this equation is contained in $B_\sharp^{\rho,M+1} \cap [-1,1]^{M+1}$. The set on the right hand side of this equation is a product set and so its volume is easily calculated. Therefore we conclude

$$Vol^{\mathbb{R}^{M+1}}\left[B_\sharp^{\rho,M+1} \cap [-1,1]^{M+1}\right] \geq Vol^{\mathbb{R}^M}\left[B_\sharp^{\rho,M} \cap [-1,1]^M\right]$$

which proves that the volume is nondecreasing in M.

Next, we calculate the volume if $M = 2$. Simplifying the constraints defining $B_\sharp^\rho \cap [-1,1]^2$, we see that it is defined by the equations

$$\begin{aligned} 0 \leq& x_2 \leq 1, \\ \max\left(-1, -\rho^{-1}x_2\right) \leq& x_1 \leq 1. \end{aligned}$$

Geometrically, this region is the union of the square $[0,1] \times [0,1]$ with a portion of the square $[-1,0] \times [0,1]$, and its volume is strictly greater than one for any $\rho \in (0,\infty)$. The exact value is

$$Vol^{\mathbb{R}^2}\left[B_\sharp^\rho \cap [-1,1]^2\right] = \begin{Bmatrix} 2 - \rho/2 & (\rho < 1) \\ 1 + \frac{1}{2\rho} & (\rho \geq 1) \end{Bmatrix}.$$

5. Sharpness

We prove Proposition 5 by constructing counterexamples. Let $\rho \in (0,\infty)$ and $M \in \mathbb{N}$ be fixed. In the following section, we will sometimes use "n" and

"$Ml+j$" interchangeably, so $x_n = j$ means $x_{Ml+j} = j$, for example. The counterexamples will have $a_n = |a_n|e^{i\theta_n}$, with $|a_n|, \theta_n$ defined as follows:

$$\begin{aligned}(|a_{Ml+1}|, \ldots, |a_{Ml+M}|) &= l^{-1}\alpha,\\ (\cos(\theta_{Ml+1}), \ldots, \cos(\theta_{Ml+M})) &= \delta + \gamma(1,1,\ldots,1),\\ \sin(\theta_{Ml+j}) &= (-1)^l\sqrt{1-\cos^2(\theta_{Ml+j})},\end{aligned} \tag{14}$$

where $\alpha \in B^\rho, \delta \in B^\rho_\sharp$ and $\gamma \geq 0$ are yet to be determined (subject to the requirement $\delta_j + \gamma \in [-1,1]$). We see that our construction already has the following properties.

- $\sum a_n n^{-s}$ has $\sigma_a = 0$. [This is due to the factor l^{-1}].
- $(|a_{Ml+1}|, |a_{Ml+2}|, \ldots, |a_{Ml+M}|) \in B^\rho$.

We now develop a sufficient condition on α, δ and γ under which $\sum a_n n^{-s}$ has a holomorphic extension past $s = 0$. We will use Dirichlet's test.

Proposition 10 (Dirichlet's Test for Convergence)**.** *Let $x_n \geq 0$ and $y_n \in \mathbb{C}$. If $x_1 \geq x_2 \geq \cdots$ and $x_n \to 0$, and $\sum_{i=1}^N y_i$ has a uniform bound for all N, then*

$$\sum_n x_n y_n$$

converges.

Proof. The proof is a standard one and applies partial summation. $\square$

We will prove both parts of Proposition 5 using the following proposition.

Proposition 11. *Let $\rho \in (0,\infty)$ and $M \in \mathbb{N}$ be fixed. Let α, δ and $\gamma \geq 0$ satisfy*

$$\sum_{j=1}^M \alpha_j\delta_j + \gamma\sum_{j=1}^M \alpha_j = 0, \tag{15}$$

$$\delta_j + \gamma \in [-1,1] \qquad \text{for all } j = 1, \ldots, M. \tag{16}$$

Then, with a_n defined as in (14), the Dirichlet series $\sum a_n n^{-s}$ has a holomorphic extension to a neighborhood of $s = 0$.

Proof. We prove $\sum a_n n^{-s}$ has a holomorphic extension past $s = 0$ by proving that $\sum a_n n^\epsilon$ converges for some $\epsilon > 0$. To prove this, the idea is to apply Dirichlet's test to

$$\sum a_n n^\epsilon = \sum (l^{-1}n^\epsilon)\alpha_j e^{i\theta_n}$$

with $x_n = l^{-1}n^\epsilon$, $y_n = \alpha_j e^{i\theta_n}$. However, $l^{-1}n^\epsilon$ does not necessarily satisfy the nonincreasing requirement, so we first compare $\sum a_n n^\epsilon$ with the similar series defined by $b_n = Mn^{-1}\alpha_j e^{i\theta_n}$, i.e.,

$$\sum b_n n^\epsilon = \sum (Mn^{-1}n^\epsilon)\alpha_j e^{i\theta_n}.$$

Let $c_n = a_n - b_n$. We have

$$|c_n| = \left(\frac{1}{l} - \frac{M}{Ml+j}\right)\alpha_j = \frac{j}{l(Ml+j)}\alpha_j = O\left(\frac{1}{n^2}\right)$$

and therefore $\sum c_n n^\epsilon$ converges for $\epsilon < 1$, so $\sum a_n n^\epsilon$ converges if $\sum b_n n^\epsilon$ does.

We apply Dirichlet's test to $\sum b_n n^\epsilon$, with

$$x_n = Mn^{-1+\epsilon}, \;\; y_n = \alpha_j e^{i\theta_n}.$$

We see that x_n is positive, nonincreasing (for $\epsilon \le 1$), and converges to 0, so we need to have a uniform bound on

$$\sum_{n=1}^{N} \alpha_j e^{i\theta_n}.$$

Since $\alpha_j, e^{i\theta_n}$ are periodic (with period $2M$ due to the $(-1)^l$ factor in $\sin(\theta)$), it suffices to have the sum over each period equal to zero. Recalling the definitions for $\cos(\theta_n)$, $\sin(\theta_n)$, the requirement is that

$$\sum_{j=1}^{M} \alpha_j\Big((\delta_j+\gamma) + i(-1)^1\sqrt{1-\cos^2(\theta_{Ml+j})}\Big)$$
$$+\sum_{j=1}^{M} \alpha_j\Big((\delta_j+\gamma) + i(-1)^2\sqrt{1-\cos^2(\theta_{Ml+j})}\Big) = 0.$$

The terms coming from $\sin(\theta)$ will add to zero because of the factor of -1, so the remaining requirement is $\sum_{j=1}^{M} \alpha_j(\delta_j+\gamma) = 0$, which is satisfied by the hypothesis, and the proof is complete. $\square$

We can now prove Proposition 5.

Proof of Proposition 5 Part (I). The values for M, ρ, and $\delta \in (\partial B_\sharp^\rho) \cap [-1,1]^M$ are given. Since $\delta \in \partial B_\sharp^\rho$, we can choose $\alpha^* \in B^\rho$ such that $\alpha^* \cdot \delta = 0$. We set $\gamma = 0$ and

- $(|a_{Ml+1}|, |a_{Ml+2}|, \dots, |a_{Ml+M}|) = l^{-1}\alpha^*$.
- $(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \dots, \cos(\theta_{Ml+M})) = \delta$.

The hypotheses of Proposition 11 are satisfied, therefore we have constructed a Dirichlet series with

- $\sum a_n n^{-s}$ has $\sigma_a = 0$.
- $(|a_{Ml+1}|, |a_{Ml+2}|, \dots, |a_{Ml+M}|) \in B^\rho$.
- $(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \dots, \cos(\theta_{Ml+M})) \in B_\sharp^\rho$.

- $\sum a_n n^{-s}$ has a holomorphic extension past $s = 0$ [by Proposition 11].

and Part (I) is proved. $\square$

Proof of Proposition 5 Part (II). Fix $M \geq 2$ and $0 < \rho' < \rho$. Here, we want to find $\{a_n\}$ and $\gamma > 0$ which satisfy

- $\sum a_n n^{-s}$ has $\sigma_a = 0$.
- $(|a_{Ml+1}|, |a_{Ml+2}|, \ldots, |a_{Ml+M}|) \in B^{\rho}$.
- $(\cos(\theta_{Ml+1}), \cos(\theta_{Ml+2}), \ldots, \cos(\theta_{Ml+M})) \in B_{\sharp}^{\rho'} + \gamma(1, 1, \ldots, 1)$.
- $\sum a_n n^{-s}$ has a holomorphic extension past $s = 0$.

To meet these four conditions, we choose $\{a_n\}$ as in (14), with $\alpha_j = \rho^{-j}$, and then it suffices to satisfy the third condition listed above and satisfy the hypotheses of Proposition 11. Recalling that $\cos(\theta_n) = \delta_j + \gamma$, the third condition becomes

$$(\delta_1, \ldots, \delta_M) + \gamma(1, 1, \ldots, 1) \in B_{\sharp}^{\rho'} + \gamma(1, 1, \ldots, 1)$$

which is

$$(\delta_1, \ldots, \delta_M) \in B_{\sharp}^{\rho'}.$$

To satisfy the hypotheses of Proposition 11, we also need to choose δ_j and $\gamma \geq 0$ such that

$$\sum_{j=1}^{M} (\rho^{-j})\delta_j + \gamma \sum_{j=1}^{M} (\rho^{-j}) = 0,$$
$$\delta_j + \gamma \in [-1, 1],$$

(conditions (15) and (16), with $\alpha_j = \rho^{-j}$).

Let us define $x = \big(-(\rho')^{-(M-1)}, 0, \ldots, 0, 1\big)$ and $\delta = \epsilon x$, where ϵ will be a small positive number. We need to show that $\delta \in B_{\sharp}^{\rho'}$ and then we need to find $\gamma > 0$ such that (15) and (16) are satisfied.

We see that $\sum_{j=1}^{M} (\rho')^{-j}\delta_j = \epsilon\left(-(\rho')^{-M} + (\rho')^{-M}\right) = 0$, and for $r > 1$ we have

$$\sum_{j=r}^{M} (\rho')^{-j}\delta_j = \epsilon(\rho')^{-M} > 0.$$

Therefore, by equation (12), $\delta \in B_{\sharp}^{\rho'}$.

Examining the first term of equation (15), we have

$$\sum_{j=1}^{M} \rho^{-j}\delta_j = \epsilon\left(-\rho^{-1}(\rho')^{-(M-1)} + \rho^{-M}\right) = \epsilon\rho^{-1}\left(\rho^{-M} - (\rho')^{-(M-1)}\right) < 0.$$

Therefore we can solve equation (15) for γ and we have

$$\gamma = -\left(\sum_{j=1}^{M} \rho^{-j}\delta_j\right) / \sum_{j=1}^{M} \rho^{-j} = -\epsilon\rho^{-1}\left(\rho^{-(M-1)} - (\rho')^{-(M-1)}\right) / \sum_{j=1}^{M} \rho^{-j}.$$

This choice of γ satisfies equation (15), we have $\gamma > 0$, and γ scales linearly in ϵ. Therefore, by letting ϵ approach zero, we will satisfy (16) as well. This completes the proof. □

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