A NOTE ON LAGRANGE'S METHOD OF VARIATION OF PARAMETERS

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Abstract. In this note we provide a geometrical interpretation for the basic assumptions made in the method of variation of parameters applied to second order ordinary differential equations. We also discuss a physical motivation drawn from celestial mechanics.

1. Introduction. The method of osculating elements popularly known as the method of variation of parameters has its origins in Celestial Mechanics. The method is credited to J. L. Lagrange who systematized it and used it extensively in his research on Celestial Mechanics, particularly the motion of comets. It is important to note, and this is not emphasized in modern texts, that Lagrange employed this method to nonlinear differential equations as well [2] and [3].

The purpose of this note is to motivate the basic assumptions, (2.3) and (2.4) below, in the case of a linear second order equation. Most modern texts justify the seemingly ad hoc hypothesis (2.4) on grounds that since two functions $c_1(t)$ and $c_2(t)$ have to be determined, it is reasonable to impose condition (2.4), noting further that this leads to the correct solution as may be verified *a posteriori.* This justification however, seems arbitrary, especially to students, and we provide here a simple geometric interpretation. These ideas are implicit in the classical works on Celestial Mechanics [1] and we hope the note will be useful in the teaching of this method to undergraduates.

2. The Method and Its Geometric Meaning.

Details of the Method. We describe briefly the method and then proceed to the motivation behind the method. We consider a second order linear differential equation

$$y'' + P(t)y' + Q(t)y = R(t). \tag{2.1}$$

Throughout this paper the coefficient functions $P(t)$, $Q(t)$, and $R(t)$ are continuous in a fixed open interval. We assume that the associated homogeneous equation

$$y'' + P(t)y' + Q(t)y = 0 \tag{2.2}$$

has been solved completely and two linearly independent solutions $y_1(t)$ and $y_2(t)$ have been found. The basic assumption in the method of variation of parameters is that a particular solution to (2.1) may be found in the form

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t). \tag{2.3}$$

Differentiating (2.3) once, postulating

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0 \tag{2.4}$$

and substituting into (2.1) we get the pair of equations

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0, \quad c_1'(t)y_1' + c_2'(t)y_2'(t) = R(t). \tag{2.5}$$

Equations (2.5) may be uniquely solved for $c_1'(t)$ and $c_2'(t)$, integrated and substituted into (2.3) to yield a particular solution of (2.1).

Physical and Geometrical Interpretations. To motivate the basic assumption (2.3) let us recall the governing equations for a planet revolving around the sun under the sun's gravitational influence. (See chapter 17 of [5] for a delightful account of the two body problem.) Denoting by $y(t)$ the position vector of the planet at time t, the differential equations of motion are

$$y'' = \frac{-ky}{|y|^3}, \tag{2.6}$$

where k is a positive constant. Here, the origin is placed at the center of mass of the planet and the sun. Since time does not enter explicitly into the system (2.6), $y(t+a)$ is a solution whenever $y(t)$ is a solution. Thus, one of the constants in the general solution may be eliminated through a time translation and the complete solution, denoted by

$$y = y(t, \mathbf{c}), \tag{2.7}$$

involves five arbitrary constants $\mathbf{c} = (c_1, c_2, \ldots, c_5)$. It is well-known [5] that the motion described by (2.7) is a conic and in fact, a fixed ellipse in the case of a planet. In the presence of a perturbing third body, the ellipse (2.7) will not be fixed but would slowly precess, causing the parameters $c_1, \ldots, c_5$ to be slowly varying functions of time. Thus, one is led to make the basic assumption

$$y = y(t, \mathbf{c}(t)) \tag{2.8}$$

for the solution of the perturbed problem. A case in point is the residual precession of $43''$ per century in the orbit of Mercury, due to relativistic correction terms to the Newtonian force field [4]. Note that (2.8) is the exact analogue of (2.3) for the system (2.6). Let us assume that at time $t = \xi$ the disturbing function is shut off, causing the original trajectory $y = y(t)$ to deviate tangentially to a new trajectory $y = Y(t, \xi)$ with

$$y(\xi) = Y(\xi), \quad y'(\xi) = \frac{dY}{dt}(t, \xi)\Big|_{t=\xi}, \tag{2.9}$$

the second condition being the tangency condition. Since the full solution to the unperturbed problem (2.6) is assumed known, it is possible to determine $Y(t,\xi)$ for $t > \xi$. The actual trajectory (2.8) may therefore be obtained as the envelope of the one parameter family of fictitious trajectories $Y(t,\xi)$.

We now illustrate this idea in the case of a linear system (2.1). We have in this case,

$$Y(t,\xi) = c_1(\xi)y_1(t) + c_2(\xi)y_2(t),$$

and so

$$Y_t(\xi,\xi) = c_1(\xi)y_1'(\xi) + c_2(\xi)y_2'(\xi).$$

Conditions (2.9) read

$$y(\xi) = c_1(\xi)y_1(\xi) + c_2(\xi)y_2(\xi), \quad y'(\xi) = c_1(\xi)y_1'(\xi) + c_2(\xi)y_2'(\xi). \quad (2.10)$$

These must hold for all values of ξ in the interval on which the differential equation is defined and uniquely solvable for $c_1(\xi)$ and $c_2(\xi)$. Moreover, $c_1(\xi)$ and $c_2(\xi)$ are both once continuously differentiable. Differentiating with respect to ξ the first of (2.10) and subtracting from the second we get

$$c_1'(\xi)y_1(\xi) + c_2'(\xi)y_2(\xi) = 0. \quad (2.11)$$

Thus, the seemingly arbitrary hypothesis (2.4) assumes a clear geometrical meaning namely, it is precisely the envelope condition $\partial Y/\partial\xi = 0$ at the point of tangency. From the ODEs (2.1) and (2.2), the accelerations satisfy

$$\lim_{t\to\xi+} (y''(t) - Y''(t)) = R(\xi),$$

that is

$$y''(\xi) - c_1(\xi)y_1''(\xi) - c_2(\xi)y_2''(\xi) = R(\xi).$$

Using the second equation in (2.10) we get

$$c_1'(\xi)y_1'(\xi) + c_2'(\xi)y_2'(\xi) = R(\xi). \quad (2.12)$$

The pair (2.11)-(2.12) determining $c_1(\xi)$ and $c_2(\xi)$ is identical to the pair (2.5).

Concluding Remarks. Note that the functions $c_1(t)$ and $c_2(t)$ in (2.3) may not only be altered by adding arbitrary constants but, due to the identity

$$c_1(t)y_1(t) + c_2(t)y_2(t) = (c_1(t) - y_2(t)\lambda(t))y_1(t) + (c_2(t) + y_1(t)\lambda(t))y_2(t),$$

may also be altered by the addition of $-y_2(t)\lambda(t)$ and $y_1(t)\lambda(t)$, respectively. If we choose

$$\lambda(t) = \frac{c_1'(t)y_1(t) + c_2'(t)y_2(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)},$$

then (2.4) is seen to hold with $c_1(t) - y_2(t)\lambda(t)$ and $c_2(t) + y_1(t)\lambda(t)$ in place of $c_1(t)$ and $c_2(t)$. We have provided a geometrical interpretation for the seemingly ad hoc hypothesis (2.4), obtaining both the equations (2.5) as a natural geometric consequence.

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References

1. D. Boccaletti and G. Puccaco, *Theory of Orbits - II, Perturbative and Geometrical Methods,* Springer Verlag, Heidelberg, Germany, 2002.
2. J. L. Lagrange, *Analytical Mechanics,* translated and edited by Auguste Boissonnade and Victor N. Vagliente, Kluwer Academic Publishers, Dordrecht, Netherlands, 1997.
3. H. Pollard, *Mathematical Introduction to Celestial Mechanics,* Prentice-Hall Inc., Englewood Cliffs, NJ, 1966.
4. N. T. Roseveare, *Mercury's Perihelion from Le Verrier to Einstein,* Clarendon Press, Oxford, England, 1982.
5. M. Spivak, *Calculus*, Third Ed., Publish or Perish Inc., Houston, TX, 1994.

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