# REDUCIBILITY AND THE GALOIS GROUP OF A PARAMETRIC FAMILY OF QUINTIC POLYNOMIALS

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**Abstract.** It is shown that $f_t(x) = x^5 + (t^2 - 3125)x - 4(t^2 - 3125)$ $(t \in \mathbb{Q})$ is reducible in $\mathbb{Q}[x]$ if and only if $t = 0$. When $t \neq 0$ it is shown that $\mathrm{Gal}(f_t) \simeq D_5$ or $A_5$, and necessary and sufficient conditions are given for each possibility.

**1. Introduction.** Smith [3] has shown that the Galois group of

$$f_t(x) = x^5 + (t^2 - 3125)(x - 4) \tag{1.1}$$

over $\mathbb{Q}(t)$ is $A_5$. Let $t \in \mathbb{Q}$. By Hilbert's irreducibility theorem for infinitely many values of $t \in \mathbb{Q}$ the polynomial $f_t(x)$ has Galois group $A_5$ over $\mathbb{Q}$. The exceptions, which occur when either the polynomial is reducible over $\mathbb{Q}$ or is irreducible over $\mathbb{Q}$ but its Galois group is not $A_5$, form a "thin" set. In this paper we determine this set for the family (1.1). We set

$$g(u) = \frac{(u^3 - 18u^2 + 8u - 16)(u^3 + 2u^2 + 18u + 4)}{2u^2(u^2 + 4)}, \quad u \in \mathbb{Q} \setminus \{0\}, \tag{1.2}$$

and prove the following result.

<u>Theorem</u>.

(a) Let $t \in \mathbb{Q}$. Then $f_t(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $t = 0$. If $t = 0$ we have

$$f_0(x) = x^5 - 3125x + 12500 = (x - 5)^2(x^3 + 10x^2 + 75x + 500).$$

(b) If $t \in \mathbb{Q} \setminus \{0\}$ then

$$\mathrm{Gal}(f_t(x)) \simeq D_5 \text{ if } t = g(u) \text{ for some } u \in \mathbb{Q} \setminus \{0\}$$

and

$$\mathrm{Gal}(f_t(x)) \simeq A_5 \text{ if } t \neq g(u) \text{ for any } u \in \mathbb{Q} \setminus \{0\}.$$

Example 1. If $t = -\frac{125}{2}$ then $t = g(1)$ and by the theorem we have

$$\mathrm{Gal}(f_{-125/2}(x)) = \mathrm{Gal}\left(x^5 + \frac{3125}{4}x - 3125\right) \simeq D_5.$$

Example 2. If $t = 1$ then as

$$(x^3 - 18x^2 + 8x - 16)(x^3 + 2x^2 + 18x + 4) - 2x^2(x^2 + 4)$$

is irreducible in $\mathbb{Q}[x]$ there does not exist $u \in \mathbb{Q}$ such that $t = g(u)$ and by the theorem

$$\mathrm{Gal}(f_1(x)) = \mathrm{Gal}(x^5 - 3124x + 12496) \simeq A_5.$$

Example 3. As

$$\lim_{u \to 0^+} g(u) = -\infty, \quad \lim_{u \to +\infty} g(u) = +\infty,$$

and $g(u)$ is strictly increasing for $u > 0$, it is clear that $g(u)$ assumes infinitely many distinct (rational) values for $u \in \mathbb{Q}^+$. Hence, by the theorem, there are infinitely many $t \in \mathbb{Q}$ for which $\mathrm{Gal}(f_t(x)) \simeq D_5$.

Example 4. Let $t = 3n$, $n \in \mathbb{N}$. Suppose there exists $u \in \mathbb{Q} \setminus \{0\}$ with $3n = g(u)$. Then the sextic polynomial

$$(x^3 - 18x^2 + 8x - 16)(x^3 + 2x^2 + 18x + 4) - 6nx^2(x^2 + 4)$$

has a rational root. However,

$$\begin{aligned} &(x^3 - 18x^2 + 8x - 16)(x^3 + 2x^2 + 18x + 4) - 6nx^2(x^2 + 4) \\ &\equiv (x^3 + 2x + 2)(x^3 + 2x^2 + 1) \pmod 3 \end{aligned}$$

has no roots $\pmod 3$. Hence, no such $u$ exists and by the theorem there exist infinitely many $t \in \mathbb{Q}$ such that $\mathrm{Gal}(f_t(x)) \simeq A_5$.

We conclude this introduction by recalling a few facts about quintic trinomials, which will be used in the proof of the Theorem in Section 2.

Proposition 1. [2] Let $A$ and $B$ be rational numbers. The discriminant of $x^5 + Ax + B$ is $4^4A^5 + 5^5B^4$.

Proposition 2. [5] Let $A$ and $B$ be rational numbers such that $4^4A^5 + 5^5B^4 > 0$. Then $x^5 + Ax + B$ has exactly one real root.

Proposition 3. [4] Let $A$ and $B$ be rational numbers such that the quintic trinomial $x^5 + Ax + B$ is irreducible in $\mathbb{Q}[x]$. Then $x^5 + Ax + B$ is solvable by radicals if and only if there exist rational numbers $\epsilon(= \pm 1)$, $C(\geq 0)$ and $E(\neq 0)$ such that

$$A = \frac{5E^4(3 - 4\epsilon C)}{C^2 + 1}, \quad B = \frac{-4E^5(11\epsilon + 2C)}{C^2 + 1}.$$

Proposition 4. [4] Let $\epsilon(= \pm 1)$, $C(\geq 0)$ and $E(\neq 0)$ be rational numbers such that the quintic trinomial

$$x^5 + \frac{5E^4(3 - 4\epsilon C)}{C^2 + 1}x - \frac{4E^5(11\epsilon + 2C)}{C^2 + 1}$$

is irreducible in $\mathbb{Q}[x]$. Then the Galois group of $x^5 + Ax + B$ is the dihedral group $D_5$ of order 10 if and only if $5(C^2 + 1)$ is a perfect square in $\mathbb{Q}$.

**2. Proof of Theorem.** (a) If $t = 0$ we have

$$f_0(x) = x^5 - 3125x + 12500 = (x - 5)^2(x^3 + 10x^2 + 75x + 500).$$

Now suppose $t \in \mathbb{Q} \backslash \{0\}$. We show that $f_t(x)$ is irreducible in $\mathbb{Q}[x]$. Suppose not. Then $f_t(x)$ has either a rational root or an irreducible quadratic factor.

Suppose first that $f_t(r) = 0$ with $r \in \mathbb{Q}$ so

$$r^5 + (t^2 - 3125)(r - 4) = 0. \tag{2.1}$$

Clearly $r \neq 4, 5$. Set

$$x = \frac{-17r - 188}{r - 4} \in \mathbb{Q} \tag{2.2}$$

and

$$y = \frac{8(r^2 + 7r + 16t - 60)}{(r - 4)(r - 5)} \in \mathbb{Q}. \tag{2.3}$$

Then

$$y^2 + xy + y - x^3 - 549x + 2202 = \frac{2^{14}(r^5 + (t^2 - 3125)(r-4))}{(r-4)^3(r-5)^2} = 0. \quad (2.4)$$

This elliptic curve is A4(H) of [1]. Its conductor is 50, its rank is 0 and the order of the torsion subgroup is 1. Thus, there are no pairs $(x, y) \in \mathbb{Q}^2$ satisfying (2.4), contradicting (2.2)–(2.4).

Now suppose $f_t(x)$ has the irreducible quadratic factor $x^2 + ax + b$ ($a, b \in \mathbb{Q}$, $a^2 - 4b \notin \mathbb{Q}^2$). As

$$\begin{aligned} &x^5 + (t^2 - 3125)x - 4(t^2 - 3125) \\ &\quad = (x^2 + ax + b)(x^3 - ax^2 + (a^2 - b)x + (2ab - a^3)) \\ &\quad\ + (a^4 - 3a^2b + b^2 + t^2 - 3125)x + (a^3b - 2ab^2 - 4t^2 + 12500) \end{aligned}$$

we must have

$$a^4 - 3a^2b + b^2 + t^2 - 3125 = a^3b - 2ab^2 - 4t^2 + 12500 = 0. \quad (2.5)$$

Eliminating $t^2$ from (2.5), we obtain

$$(4 - 2a)b^2 + (a^3 - 12a^2)b + 4a^4 = 0. \quad (2.6)$$

If $a = -10$ then $b = 25$ or $200/3$ so $t^2 = 0$ or $78125/9$, a contradiction. If $a = 0$ then $b = 0$ and $t^2 = 3125$, a contradiction. If $a = 2$ then $b = 8/5$ and $t^2 = 78141/25$, a contradiction. Hence, $a \neq -10, 0, 2$. Solving the quadratic equation (2.6) for $b$ we obtain

$$b = \frac{12a^2 - a^3 \pm a^2\sqrt{a^2 + 8a + 80}}{8 - 4a}. \quad (2.7)$$

As $b \in \mathbb{Q}$ there exists $z \in \mathbb{Q}$ such that

$$a^2 + 8a + 80 = z^2. \quad (2.8)$$

Hence,

$$(z + a + 4)(z - a - 4) = 64.$$

Thus, there exists $k \in \mathbb{Q} \setminus \{0\}$ such that

$$z + a + 4 = k, \quad z - a - 4 = \frac{64}{k}. \tag{2.9}$$

Solving (2.9) for $a$ and $z$, we obtain

$$a = \frac{k^2 - 8k - 64}{2k}, \quad z = \frac{k^2 + 64}{2k}. \tag{2.10}$$

As $a \neq 2$ we have $k \neq -4, 16$. As $a \neq -10$ we have $k \neq 4, -16$. Hence, $k \neq 0, \pm 4, \pm 16$. Using (2.10) in (2.7) we deduce $b = b_1$ or $b_2$, where

$$b_1 = \frac{k^4 - 16k^3 - 64k^2 + 1024k + 4096}{8k^2 + 32k}, \tag{2.11}$$

$$b_2 = \frac{-2k^4 + 32k^3 + 128k^2 - 2048k - 8192}{k^3 - 16k^2}. \tag{2.12}$$

First, using the values of $a$ and $b_1$ in (2.5), we find

$$t^2 = \frac{(k-4)(k^3 - 52k^2 + 768k + 4096)(k^3 + 8k^2 + 88k + 256)^2}{64k^4(k+4)^2}. \tag{2.13}$$

Set

$$x = \frac{2(k+46)}{k-4} \in \mathbb{Q}, \tag{2.14}$$

$$y = \frac{-100k^2(k+4)t}{(k-4)^2(k^3 + 8k^2 + 88k + 256)} - \frac{(3k+88)}{2(k-4)} \in \mathbb{Q}. \tag{2.15}$$

Then

$$\frac{2^2}{5^4}(y^2 + yx + y - x^3 + 76x - 298) =$$

$$\frac{64k^4(k+4)^2t^2 - (k-4)(k^3 - 52k^2 + 768k + 4096)(k^3 + 8k^2 + 88k + 256)^2}{(k-4)^4(k^3 + 8k^2 + 88k + 256)^2}.$$

Thus, by (2.13), we have

$$y^2 + yx + y - x^3 + 76x - 298 = 0. \tag{2.16}$$

The elliptic curve (2.16) is curve A3(G) [1]. The conductor is 50, the rank is 0 and the order of the torison subgroup is 3. There are exactly two finite rational points on this curve, namely, $(2, 11)$ and $(2, -14)$. It is clear from (2.14) that these do not correspond to a rational value of $k$.

Next, by using the values of $a$ and $b_2$ in (2.5), we obtain

$$t^2 = \frac{-(k+16)(k^3 - 12k^2 - 52k - 64)(k^3 - 22k^2 + 128k - 1024)^2}{16k^4(k-16)^2}. \tag{2.17}$$

As $k \neq 0$ we can set $k_1 = -64/k \in \mathbb{Q} \setminus \{0\}$. As $k \neq \pm 4, \pm 16$ we have $k_1 \neq \pm 4, \pm 16$. Replacing $k$ by $-64/k_1$ in (2.17), we obtain (2.13) with $k$ replaced by $k_1$, which we have shown has no rational solutions $(t, k_1)$ with $k_1 \neq 0, \pm 4, \pm 16$.

This completes the proof of part (a) of the theorem.

(b) We now turn to the proof of part (b). Let $t \in \mathbb{Q} \setminus \{0\}$. By Proposition 1 the discriminant of $f_t(x)$ is $2^8 t^2 (t^2 - 3125)^4$. As the discriminant $\in \mathbb{Q}^2$, $\mathrm{Gal}(f_t(x))$ is isomorphic to one of $\mathbb{Z}_5$, $D_5$ or $A_5$. It is easy to see by Rolle's Theorem that $f_t(x)$ has at most three real roots (indeed by Proposition 2 it has exactly one real root) so $\mathrm{Gal}(f_t(x)) \not\simeq \mathbb{Z}_5$. Thus, $\mathrm{Gal}(f_t(x)) \simeq D_5$ or $A_5$.

Suppose first that there exists $u \in \mathbb{Q} \setminus \{0\}$ such that $t = g(u)$, where $g$ is defined in (1.2). Set

$$c = \left| \frac{11u^2 + 8u - 44}{2u^2 - 44u - 8} \right| \in \mathbb{Q}, \tag{2.18}$$

$$e = \left( \operatorname{sgn} \left( \frac{11u^2 + 8u - 44}{2u^2 - 44u - 8} \right) \right) \frac{(u^2 - 2u - 4)}{2u} \in \mathbb{Q}, \tag{2.19}$$

$$\epsilon = -\operatorname{sgn} \left( \frac{11u^2 + 8u - 44}{2u^2 - 44u - 8} \right) = \pm 1. \tag{2.20}$$

We note that $c \geq 0$ and $e \neq 0$. Then

$$t^2 - 3125 = \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1}$$

and

$$-4(t^2-3125)=\frac{-4e^5(11\epsilon+2c)}{c^2+1}$$

so

$$f_t(x)=x^5+\frac{5e^4(3-4\epsilon c)}{c^2+1}x-\frac{4e^5(11\epsilon+2c)}{c^2+1}.$$

Further

$$5(c^2+1)=\left(\frac{25(u^2+4)}{2(u^2-22u-4)}\right)^2\in\mathbb{Q}^2$$

so by Proposition 4, $\mathrm{Gal}(f_t)\simeq D_5$.

Conversely, suppose that $\mathrm{Gal}(f_t(x))\simeq D_5$. Hence, $f_t(x)=0$ is solvable by radicals. Then, by Propostion 3, there exist rationals $c(\geq 0)$, $\epsilon(=\pm 1)$ and $e(\neq 0)$ such that

$$t^2-3125=\frac{5e^4(3-4\epsilon c)}{c^2+1},\quad -4(t^2-3125)=\frac{-4e^5(11\epsilon+2c)}{c^2+1}. \tag{2.21}$$

Eliminating $t^2-3125$, we obtain

$$c=\frac{15-11\epsilon e}{2(e+10\epsilon)}. \tag{2.22}$$

Then, from (2.22) and the first equation in (2.21), we deduce

$$t^2=\frac{(2e^3+10\epsilon e^2-25e+125\epsilon)^2}{(e^2-2\epsilon e+5)}. \tag{2.23}$$

From (2.23) we see that there exists $z\in\mathbb{Q}\setminus\{0\}$ such that

$$e^2-2\epsilon e+5=z^2.$$

Hence,

$$(z-e+\epsilon)(z+e-\epsilon)=4.$$

Thus, there exists $u \in \mathbb{Q} \setminus \{0\}$ such that

$$z + e - \epsilon = -\epsilon u,$$

$$z - e + \epsilon = -\frac{4\epsilon}{u}.$$

Solving these two equations for $e$ we find

$$e = -\epsilon \left( \frac{u^2 - 2u - 4}{2u} \right). \tag{2.24}$$

From (2.23) and (2.24) we obtain

$$t^2 = \frac{(u^3 - 18u^2 + 8u - 16)^2 (u^3 + 2u^2 + 18u + 4)^2}{4u^4 (u^2 + 4)^2}$$

so that

$$t = \pm g(u).$$

If the plus sign holds then $t = g(u)$ as required. If the minus sign holds then $t = -g(u) = g(-4/u)$ as required.

This completes the proof of the theorem.

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