A NOTE ON PARAQUATERNIONIC MANIFOLDS

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Abstract. In this paper we show that a paraquaternionic nearly Kähler manifold is necessarily a paraquaternionic Kähler manifold.

1. Introduction. The theory of paraquaternionic manifolds has been developing in recent years, [1, 4, 6, 7, 8, 12], although its roots date back to the 1950's in the work of P. Libermann [9]. The concept of quaternionic structure of the second kind has been defined as a triplet of endomorphisms of the tangent bundle $\{J_1, J_2, J_3\}$, in which J_1 is almost complex and J_2 and J_3 are almost product structures satisfying some relations of anti-commutation. The differential geometry of spaces having such a structure is interesting. For example, S. Ianuş [5] showed the existence of such a structure on the tangent bundle of a manifold (without the condition that the dimension of the underlying manifold be a multiple of 4). Moreover, an integrable paraquaternionic structure appeared naturally in the study of Osserman pseudo-Riemannian manifolds [3]. Under certain conditions on the holonomy group of a metric adapted to such a structure, one arrives at the concept of paraquaternionic Kähler manifold. An important example in this theory is the paraquaternionic projective space as described by Blazic [2].

In this note we give some details on paraquaternionic nearly Kählerian manifolds, which turn out to be paraquaternionic Kählerian manifolds.

This might be surprising since in complex geometry the class of nearly Kähler manifolds is interesting in its own right and contains as a non-trivial subclass the Kählerian ones [10].

2. Preliminaries. Let M be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ of sections of σ exists satisfying

$$\begin{cases} J_\alpha^2 = -\epsilon_\alpha Id \\ J_1 J_2 = -J_2 J_1 = J_3, \end{cases} \tag{1}$$

where $\alpha = 1, 2, 3$, $\epsilon_1 = 1$, $\epsilon_2 = \epsilon_3 = -1$.

Then the bundle σ is called a paraquaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ. Moreover, (M, σ) is said to be an almost paraquaternionic manifold.

In an almost paraquaternionic manifold (M, σ) we take intersecting coordinate neighborhoods U and U'. Let $\{J_1, J_2, J_3\}$ and $\{J'_1, J'_2, J'_3\}$ be canonical local bases of σ in U and U', respectively. Then $\{J'_1, J'_2, J'_3\}$ are linear combinations of $\{J_1, J_2, J_3\}$ in $U \cap U'$:

$$\begin{cases} J'_1 = a_{11}J_1 + a_{12}J_2 + a_{13}J_3 \\ J'_2 = a_{21}J_1 + a_{22}J_2 + a_{23}J_3 \\ J'_3 = a_{31}J_1 + a_{32}J_2 + a_{33}J_3, \end{cases} \tag{2}$$

where $a_{\alpha\beta}$ are functions in $U \cap U'$, $\alpha, \beta = 1, 2, 3$ and $A = (a_{\alpha\beta})_{\alpha,\beta=1,2,3} \in SO(2,1)$.

A pseudo-Riemannian metric g is said to be adapted to the paraquaternionic structure σ if it satisfies

$$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y), \ \alpha = 1, 2, 3 \tag{3}$$

for all vector fields X, Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ, which also gives that J_α are skew-symmetric with respect to g. Moreover, (M, σ, g) is said to be a paraquaternionic manifold.

It is clear that any paraquaternionic manifold is of dimension $n = 4m$ and any adapted metric is necessarily of neutral signature $(2m, 2m)$.

If the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g, then (M, σ, g) is said to be a paraquaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that

$$\begin{cases} \nabla_X J_1 = -\omega_3(X)J_2 + \omega_2(X)J_3 \\ \nabla_X J_2 = -\omega_3(X)J_1 + \omega_1(X)J_3 \\ \nabla_X J_3 = \quad \omega_2(X)J_1 - \omega_1(X)J_2 \end{cases} \tag{4}$$

for any vector field X on M.

Let (M, σ, g) be a paraquaternionic manifold. If $X \in T_pM$ and $p \in M$, then the 4-plane $PQ(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a paraquaternionic 4-plane. A 2-plane in T_pM spanned by $\{X, Y\}$ is called half-paraquaternionic if $PQ(X) = PQ(Y)$.

The sectional curvature for a half-paraquaternionic 2-plane is called a paraquaternionic sectional curvature. A paraquaternionic Kähler manifold of constant paraquaternionic sectional curvature is said to be a paraquaternionic space form.

Let $\{J_1, J_2, J_3\}$ be a canonical local base of σ in a coordinate neighborhood U of a paraquaternionic manifold (M, σ, g). If we denote by

$$\Omega_{J_\alpha}(X, Y) = g(X, J_\alpha Y), \ \alpha = 1, 2, 3 \tag{5}$$

for any vector fields X and Y, then by means of (2), we see that

$$\Omega = \Omega_{J_1} \wedge \Omega_{J_1} - \Omega_{J_2} \wedge \Omega_{J_2} - \Omega_{J_3} \wedge \Omega_{J_3} \tag{6}$$

is a globally well-defined 4-form on M.

We say that a paraquaternionic manifold (M, σ, g) is a nearly-paraquaternionic Kähler manifold if and only if the Levi-Civita connection of g satisfies

$$(\nabla_X \Omega)(X, Y, Z, W) = 0, \tag{7}$$

where X, Y, Z, W are sections in the tangent bundle of M. The main purpose of this note is to prove the following result.

Theorem 2.1. Let (M, σ, g) be a nearly-paraquaternionic Kähler manifold with $\dim M \geq 8$. Then M is a paraquaternionic Kähler manifold.

3. Proof of Theorem. In order to prove the theorem, we first need the following lemma.

Lemma 3.1. Let (M, g) be a pseudo-Riemannian manifold and a skew-symmetric endomorphism J of TM. Then we have

i.

$$(\Omega_J \wedge \Omega_J)(X, Y, Z, W) = 2 \sum_{Y,Z,W} \Omega_J(X, Y)\Omega_J(Z, W), \tag{8}$$

where the sum is taken over cyclic permutations of Y,Z,W.

ii.

$$\begin{aligned} &\nabla_U(\Omega_J \wedge \Omega_J)(X, Y, Z, W) \\ &= 2 \sum_{Y,Z,W} g(X, (\nabla_U J)Y)\Omega_J(Z, W) + \Omega_J(X, Y)g(Z, (\nabla_U J)W), \end{aligned} \tag{9}$$

where the sum is taken over cyclic permutations of Y,Z,W.

iii.

$$g((\nabla_X J)Y, Z) + g((\nabla_X J)Z, Y)) = 0. \tag{10}$$

Moreover if $J^2 = \pm Id$, then

iv.

$$(\nabla_X J)(JY) = -J((\nabla_X J)Y), g((\nabla_X J)Y, JY) = g((\nabla_X J)Y, Y) = 0. \tag{11}$$

v. $(\nabla_X J)(Y) = 0$ for any pair of Hermitian totally real vectors X and Y (i.e. $Y \perp Span\{X, JX\}$ if and only if M is Kähler (i.e. $(\nabla_X J)(Y) = 0$).

Furthermore, if J_1, J_2, J_3 satisfy the paraquaternionic identities (1), then we have that vi.

$$(\nabla_X J_1)(J_2 Y) = (\nabla_X J_3)Y - J_1(\nabla_X J_2)Y,$$
$$(\nabla_X J_2)(J_1 Y) = -(\nabla_X J_3)Y - J_2(\nabla_X J_1)Y, \tag{12}$$

and four other similar relations hold.

Proof. These are straightforward; or one may take into consideration that these tensor equations can be proven on single vectors.

One must now show that for any vectors X, Y it holds that $(\nabla_X J_\alpha)Y \in Span\{J_1 Y, J_2 Y, J_3 Y\},\ \alpha = 1, 2, 3$. This will be done by the systematic use of Lemma 3.1.

Claim 1. For paraquaternionic totally real pair of vectors X, Y (i.e. $PQ(X) \perp PQ(Y)$, or equivalently, $PQ(X) \perp Y$), one has $(\nabla_X J_\alpha)Y \in PQ(X) + PQ(Y),\ \alpha = 1, 2, 3$.

Clearly this is true if $\dim M = 8$. Otherwise, for any triple $\{X, Y, Z\}$ of pairwise paraquaternionic totally real vectors, Lemma 3.1 gives

$$(\nabla_X \Omega)(X, Y, Z, J_1 X) = -2\|X\|^2 g(Y, (\nabla_X J_1)Z)$$

$$= 2\,\|X\|^2\, g\left(\left(\nabla_X J_1\right)Y, Z\right) = 0 \tag{13}$$

and similarly for J_2 and J_3. Since M is a nearly-paraquaternionic Kähler manifold, the claim follows.

Claim 2. For paraquaternionic totally real pair of vectors X, Y one has $(\nabla_X J_\alpha)Y \perp PQ(X),\ \alpha = 1, 2, 3$.

Applying Lemma 3.1 again we obtain

$$(\nabla_X \Omega)(X, Y, J_2 X, J_1 X) = -2\|X\|^2\{g(Y, (\nabla_X J_1)(J_2 X))$$
$$+ g(J_1 X, (\nabla_X J_2)Y) - g(X, (\nabla_X J_3)Y)\}. \tag{14}$$

If we denote
$$(\nabla_X J_\alpha)Y = M_{J_\alpha}(X, Y),\ \alpha = 1, 2, 3, \tag{15}$$
we deduce from (14) and (15) that

$$(M_{J_3} + J_1 M_{J_2} - J_2 M_{J_1})(X, Y) \perp X. \tag{16}$$

Replacing Y by $J_3 Y$ and applying Lemma 3.1 gives

$$(-J_1 M_{J_1} + J_2 M_{J_2} - 3 J_3 M_{J_3})(X, Y) \perp X. \tag{17}$$

Similarly we find that

$$(M_{J_1} + J_3 M_{J_2} - J_2 M_{J_3})(X, Y) \perp X, \tag{18}$$

$$(3J_1 M_{J_1} + J_2 M_{J_2} + J_3 M_{J_3})(X, Y) \perp X, \tag{19}$$

$$(M_{J_2} + J_3 M_{J_1} - J_1 M_{J_3})(X, Y) \perp X, \tag{20}$$

$$(-J_1 M_{J_1} - 3J_2 M_{J_2} + J_3 M_{J_3})(X, Y) \perp X. \tag{21}$$

From (17), (19), (21) we see that $J_\alpha M_{J_\alpha}(X, Y) \perp X,\ \alpha = 1, 2, 3$, is equivalent to the Claim, by use of Lemma 3.1 and substitutions Y to $J_\alpha Y$.

Now we have $(\nabla_X J_\alpha)Y \in Span\{J_1 Y, J_2 Y, J_3 Y\},\ \alpha = 1, 2, 3$. Finally, taking into account (11), we obtain the assertion of our theorem.

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