THE COMPOUND F DISTRIBUTION

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Abstract. The F distributions are becoming increasingly prominent in several applied areas. In this paper, a new generalization of the F distribution is introduced by compounding. It takes the form of the product of two F pdf's. Various structural properties of this distribution are derived, including its cdf, moments, mean deviation about the mean, mean deviation about the median, entropy, asymptotic distribution of the extreme order statistics, maximum likelihood estimates and the Fisher information matrix.

1. Introduction. The F distribution is one of the most familiar statistical distributions in finance, economics and related areas [2]. The increasing applications of this distribution to income data have forced the need for more generalizations of the F distribution. The simplest form of the F distribution is given by the probability density function (pdf)

$$f(x) \propto \frac{x^{\delta-1}}{(x+y)^{\gamma}} \tag{1}$$

for $x > 0$, $y > 0$, $\gamma > 0$ and $\delta > 0$. The parameter y is the cut-off point and γ and δ describe the tail of the distribution. Usually, the exact value of y will be unknown but the experimenter will have some idea about the range of possible values of y. The only standard model for data in a finite range is the beta distribution and so it is reasonable to assume that y is distributed with the pdf

$$f(y) \propto (y-d)^{a}(c-y)^{b} \tag{2}$$

for $d < y < c$, $a > -1$ and $b > -1$. Combining (1) and (2), we obtain the compound pdf of x as

$$\begin{aligned} f(x) &\propto \int_d^c \frac{x^{\delta-1}(y-d)^{a}(c-y)^{b}}{(x+y)^{\gamma}} dy \\ &\propto \frac{x^{\delta-1}}{(x+c)^{a+1}(x+d)^{b+1}}, \end{aligned} \tag{3}$$

where we have assumed that $\gamma = a + b + 2$, i.e. the shape parameter of (1) is equal to the sum of the shape parameters of (2). The compound pdf in (3) takes the form of the product of two pdfs of the form (1). Another motivation for considering (3) as a new distribution is that products of densities often arise as the posterior in Bayesian regression analysis [8]. Thus, it is important that (3) is introduced as a new distribution and its properties studied comprehensively. Formally, let us write the compound pdf as

$$f(x) = Cx^{\alpha+a-2}(1 + cx)^{-(\alpha+\beta)}(1 + dx)^{-(a+b)} \tag{4}$$

for $x > 0$, $a > 0$, $b > 0$, $c > 0$, $d > 0$, $\alpha > 0$ and $\beta > 0$, where C denotes the normalizing constant to be determined later. We refer to (4) as the *compound F distribution*. Like the F pdf, this pdf is unimodal with its mode given by the positive root of the quadratic equation

$$-cd(b+\beta+2)x^2+\{(c + d)(\alpha + a - 2) - c(\alpha + \beta) - d(a + b)\} x+\alpha+a-2 = 0.$$

The F pdf arises as the particular case of (4) for $c = d$. Figure 1 below illustrates possible shapes of (4) for selected values of a, b, α and β. Note that the y-axes are plotted on log scale. The effect of the parameters is evident. (See Figure 1.)

In the rest of this paper, we derive various structural properties of (4), including its cdf, moments, mean deviation about the mean, mean deviation about the median, entropy, asymptotic distribution of the extreme order statistics, maximum likelihood estimates and the Fisher information matrix (FIM). These quantities play a significant role in statistics and many other areas of science. For instance, the FIM plays a key role in the analysis and applications of statistical image reconstruction methods based on Poisson data models. The elements of the FIM are a function of the reciprocal of the mean values of sinogram elements [4]. The calculation of the FIM is also of central importance in many practical systems which can be described as the output of a multidimensional linear separable-denominator system with Gaussian measurement noise, e.g. nuclear magnetic resonance (NMR) spectroscopy [5].

The calculations of this paper involve several special functions, including the Appell function of the first kind defined by

$$F_1\left(a, b, c; d; x, y\right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n x^m y^n}{(d)_{m+n} m!n!},$$

the Gauss hypergeometric function defined by

$${}_2F_1\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{x^k}{k!},$$

the Legendre function of the first kind defined by

$$P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-x}{2}\right)$$

and, the Legendre function of the second kind defined by

$$Q_\nu^\mu(x) = \frac{\sqrt{\pi}\exp(i\mu\pi)\,\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma(\nu+3/2)} x^{-\mu-\nu-1}\left(x^2-1\right)^{\mu/2}$$

$$\times\ {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \nu+\frac{3}{2}; \frac{1}{x^2}\right),$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1. [6] For $\alpha > 0$ and $\beta > 0$,

$$\int_0^\infty x^{\alpha-1}(a-x)^{\beta-1}(x+z)^{-\rho}dx$$

$$= a^{\alpha+\beta-1}z^{-\rho}B(\alpha,\beta)\ {}_2F_1\left(\alpha,\rho;\alpha+\beta;-\frac{a}{z}\right).$$

Lemma 2. [6] For $0 < \alpha < \rho + \lambda$,

$$\int_0^\infty x^{\alpha-1}(x+y)^{-\rho}(x+z)^{-\lambda}dx$$

$$= z^{-\lambda}y^{\alpha-\rho}B(\alpha,\rho+\lambda-\alpha)\ {}_2F_1\left(\alpha,\lambda;\rho+\lambda;1-\frac{y}{z}\right).$$

Lemma 3. [6] For $a > 0$, $\alpha > 0$ and $\beta > 0$,

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(1-ux)^{-\rho}(1-vx)^{-\lambda}dx$$

$$= a^{\alpha+\beta-1}B(\alpha,\beta)F_1(\alpha,\rho,\lambda,\alpha+\beta;ua,va).$$

Further properties of the above special functions can be found in [1] and [6]. The use of special functions is an important development in statistics because it allows one to simplify various expressions and their numerical computation. For instance, the Appell function of the first kind can be computed using the function `AppellF1` in Mathematica. The `hypergeom` function in Maple can be used to compute the Gauss hypergeometric function. Without these special functions, one would have to write computer codes for calculating the infinite sums and this can be time-consuming and error-prone.

2. Cumulative Distribution Function. The cdf corresponding to (4) can be calculated as

$$\begin{aligned} F(x) &= C \int_0^x y^{\alpha+a-2}(1+cy)^{-(\alpha+\beta)}(1+dy)^{-(a+b)} dy \\ &= \frac{C x^{\alpha+a-1}}{\alpha+a-1} F_1\left(\alpha+a-1, \alpha+\beta, a+b, \alpha+a; -cx, -dx\right), \qquad (5) \end{aligned}$$

which follows by an easy application of Lemma 3.

3. Moments. Suppose X is a random variable with pdf (4). Its nth moment can be expressed as

$$E\left(X^n\right) = C \int_0^\infty x^{n+\alpha+a-2}(1+cx)^{-(\alpha+\beta)}(1+dx)^{-(a+b)} dx.$$

Applying Lemma 2 to calculate the integral in (6), one obtains

$$\begin{aligned} E\left(X^n\right) = {} & C c^{1-n-\alpha-a} B(n+\alpha+a-1, \beta+b-n+1) \\ & \times \ {}_2F_1\left(n+\alpha+a-1, a+b; \alpha+\beta+a+b; 1-\frac{d}{c}\right) \qquad (7) \end{aligned}$$

for $n < 1+\beta+b$. Using special properties of the Gauss hypergeometric function, the following simpler forms for (7) can be obtained. First, the normalizing constant C in (4) is given by

$$\begin{aligned} \frac{1}{C} = {} & c^{1-\alpha-a} B(\alpha+a-1, \beta+b+1) \\ & \times \ {}_2F_1\left(\alpha+a-1, a+b; \alpha+\beta+a+b; 1-\frac{d}{c}\right). \end{aligned}$$

Second, if $\alpha = a$ and $\beta = b$ then (7) can be reduced to one of the following equivalent forms

$$E\left(X^n\right) = \frac{C 2^{2a+2b-1} \Gamma\left(a+b+1/2\right) \Gamma\left(n+2a-1\right) \Gamma\left(2b-n+1\right)}{c^{n+2a-5/4} d^{1/4} \Gamma\left(2a+2b\right)}$$

$$\times \left(1 - \frac{d}{c}\right)^{1/2-a-b} \times P_{n+a-b-3/2}^{1/2-a-b} \left(\frac{1+d/c}{2\sqrt{d/c}}\right),$$

$$E\left(X^n\right) = \frac{C 4^{a+b} \Gamma\left(a+b+1/2\right) \Gamma\left(n+2a-1\right)}{\sqrt{\pi} c^{(3a+b+n-1)/2} d^{(n+a-b-1)/2} \Gamma\left(2a+2b\right)} \left(1 - \frac{d}{c}\right)^{-(a+b)}$$

$$\times \exp\left\{-i\pi(b-a-n+1)\right\} Q_{a+b-1}^{b-a-n+1} \left(\frac{1+d/c}{1-d/c}\right),$$

$$E\left(X^n\right) = \frac{C 4^{a+b} \Gamma\left(a+b+1/2\right) \Gamma\left(2b-n+1\right)}{\sqrt{\pi} c^{(3a+b+n-1)/2} d^{(n+a-b-1)/2} \Gamma\left(2a+2b\right)} \left(\frac{d}{c} - 1\right)^{-(a+b)}$$

$$\times \exp\left\{i\pi(b-a-n+1)\right\} Q_{a+b-1}^{a+n-b-1} \left(-\frac{1+d/c}{1-d/c}\right)$$

for $n < 1 + 2b$. Finally, if $c = d$ then (7) can be reduced to the familiar form

$$E\left(X^n\right) = C c^{1-n-\alpha-a} B(n+\alpha+a-1, \beta+b-n+1)$$

for $n < 1 + \beta + b$. The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median – defined by

$$\delta_1(X) = \int_0^\infty |x - \mu| \, f(x) dx$$

and

$$\delta_2(X) = \int_0^\infty |x - M| \, f(x) dx,$$

respectively, where $\mu = \mathrm{E}(X)$ and $M = \mathrm{Median}(X)$. These measures can be calculated using the relationships

$$
\begin{aligned}
\delta_1(X) &= \int_0^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x - \mu) f(x) dx \\
&= 2 \int_{\mu}^{\infty} (x - \mu) f(x) dx \\
&= 2 \int_{\mu}^{\infty} x f(x) dx - 2\mu \left\{ 1 - F(\mu) \right\} \\
&= 2\mathrm{E}(X) - 2 \int_0^{\mu} x f(x) dx - 2\mu \left\{ 1 - F(\mu) \right\} \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2(X) &= \int_0^{M} (M - x) f(x) dx + \int_{M}^{\infty} (x - M) f(x) dx \\
&= M F(M) - M \left\{ 1 - F(M) \right\} - \int_0^{M} x f(x) dx + \int_{M}^{\infty} x f(x) dx \\
&= 2 \int_0^{M} x f(x) dx - \mathrm{E}(X). \qquad (9)
\end{aligned}
$$

The expressions for $\mathrm{E}(X)$, $F(\mu)$ and $F(M)$ are given by (5) and (7). Thus, calculating $\delta_1(X)$ and $\delta_2(X)$ amounts to calculating

$$
\int_0^{\mu} x f(x) dx \text{ and } \int_0^{M} x f(x) dx.
$$

Applying Lemma 3, it is easily seen that

$$
\begin{aligned}
\int_0^{y} x f(x) dx &= C \int_0^{y} x^{\alpha + a - 1} (1 + cx)^{-(\alpha + \beta)} (1 + dx)^{-(a + b)} dx \\
&= \frac{C y^{\alpha + a}}{\alpha + a} F_1 \left(\alpha + a, \alpha + \beta, a + b, \alpha + a + 1; -cy, -dy \right). \qquad (10)
\end{aligned}
$$

Expressions for the mean deviations follow by substituting (10) into (8) and (9).

4. Rényi Entropy. An entropy of a random variable X is a measure of variation of the uncertainty. Rényi entropy is defined by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int f^\gamma(x) dx \right\},$$

where $\gamma > 0$ and $\gamma \neq 1$ [7]. It follows easily by application of Lemma 2 that

$$\int_0^\infty f^\gamma(x)dx = C^\gamma \int_0^\infty x^{\gamma(\alpha+a-2)}(1+cx)^{-\gamma(\alpha+\beta)}(1+dx)^{-\gamma(a+b)}dx$$

$$= C^\gamma c^{\gamma-\alpha-a\gamma} B(\alpha+a\gamma-\gamma, \beta+b\gamma+\gamma)$$

$$\times \ {}_2F_1\left(\alpha+a\gamma-\gamma, a\gamma+b\gamma; \alpha+\beta+a\gamma+b\gamma; 1-\frac{\beta\gamma+\gamma-1}{\alpha\gamma-\gamma+1}\right).$$

Thus, Rényi entropy for (4) is given by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma}\Bigg\{\gamma \log C + (\gamma-\alpha-a\gamma)\log c$$

$$+ \log B(\alpha+a\gamma-\gamma, \beta+b\gamma+\gamma)$$

$$+ \log \ {}_2F_1\left(\alpha+a\gamma-\gamma, a\gamma+b\gamma; \alpha+\beta+a\gamma+b\gamma; 1-\frac{\beta\gamma+\gamma-1}{\alpha\gamma-\gamma+1}\right)\Bigg\}.$$

5. Asymptotics. If $X_1, \ldots, X_n$ is a random sample from (4) and if $\bar{X} = (X_1 + \cdots + X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\bar{X} - \mathrm{E}(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$. Note from (4) that $f(t) \sim C/\{c^{\alpha+\beta}d^{a+b}\}t^{-(\beta+b+2)}$ as

$t \to \infty$ and $f(t) \to C t^{\alpha+a-2}$ as $t \to 0$. Thus, it follows by using L'Hospital's rule that

$$\frac{1 - F(tx)}{1 - F(t)} \to x^{-(\beta+b+1)}$$

as $t \to \infty$ and

$$\frac{F(tx)}{F(t)} \to x^{\alpha+a-1}$$

as $t \to 0$. Hence, it follows from Theorem 1.6.2 in [3] that there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$\Pr\left\{a_n\left(M_n - b_n\right) \leq x\right\} \to \exp\left\{-x^{-(\beta+b+1)}\right\}$$

and

$$\Pr\left\{c_n\left(m_n - d_n\right) \leq x\right\} \to 1 - \exp\left\{-x^{\alpha+a-1}\right\}$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in [3], one can see that $b_n = 0$ and a_n satisfies $1 - F(a_n) \sim 1/n$ as $n \to \infty$. Using the fact that $1 - F(t) \sim (C/(\beta+b+1))c^{-(\alpha+\beta)}d^{-(a+b)}t^{-(\beta+b+1)}$ as $t \to \infty$, one can show that

$$a_n = \left\{\frac{nC}{(\beta+b+1)c^{\alpha+\beta}d^{a+b}}\right\}^{1/(\beta+b+1)}$$

satisfies $1 - F(a_n) \sim 1/n$. The constants c_n and d_n can be determined by using the same corollary.

6. Estimation. Here, we consider maximum likelihood estimation of the parameters when $X_1, \ldots, X_n$ is a random sample from (4) and also provide expressions for the associated FIM. The log-likelihood is

$$\log L\left(a, b, c, d, \alpha, \beta\right) = n \log C + (\alpha + a - 2) \sum_{j=1}^{n} \log X_j$$

$$- (\alpha + \beta) \sum_{j=1}^{n} \log\left(1 + cX_j\right) - (a + b) \sum_{j=1}^{n} \log\left(1 + dX_j\right).$$

The first derivatives with respect to the six parameters are:

$$\frac{\partial \log L}{\partial a} = \frac{n}{C}\frac{\partial C}{\partial a} + \sum_{j=1}^{n} \log X_j - \sum_{j=1}^{n} \log\left(1 + dX_j\right),$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{C}\frac{\partial C}{\partial b} + \sum_{j=1}^{n} \log\left(1 + dX_j\right),$$

$$\frac{\partial \log L}{\partial c} = \frac{n}{C}\frac{\partial C}{\partial c} - (\alpha + \beta) \sum_{j=1}^{n} \frac{X_j}{1 + cX_j},$$

$$\frac{\partial \log L}{\partial d} = \frac{n}{C}\frac{\partial C}{\partial d} - (a + b) \sum_{j=1}^{n} \frac{X_j}{1 + dX_j},$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{C}\frac{\partial C}{\partial \alpha} + \sum_{j=1}^{n} \log X_j - \sum_{j=1}^{n} \log\left(1 + cX_j\right)$$

and

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{C}\frac{\partial C}{\partial \beta} - \sum_{j=1}^{n} \log\left(1 + cX_j\right).$$

Thus, the maximum likelihood estimates of the six parameters are the solutions of the equations:

$$\frac{n}{C}\frac{\partial C}{\partial a} = -\sum_{j=1}^{n} \log X_j + \sum_{j=1}^{n} \log\left(1 + dX_j\right),$$

$$\frac{n}{C}\frac{\partial C}{\partial b} = -\sum_{j=1}^{n} \log\left(1 + dX_j\right),$$

$$\frac{n}{C}\frac{\partial C}{\partial c} = (\alpha + \beta) \sum_{j=1}^{n} \frac{X_j}{1 + cX_j},$$

$$\frac{n}{C}\frac{\partial C}{\partial d} = (a+b)\sum_{j=1}^{n}\frac{X_j}{1+dX_j},$$

$$\frac{n}{C}\frac{\partial C}{\partial \alpha} = -\sum_{j=1}^{n}\log X_j + \sum_{j=1}^{n}\log\left(1+cX_j\right)$$

and

$$\frac{n}{C}\frac{\partial C}{\partial \beta} = \sum_{j=1}^{n}\log\left(1+cX_j\right).$$

Calculation of the associated FIM requires second-order derivatives of $\log L$. All of the second-order derivatives take the form

$$\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j} = -\frac{n}{C^2}\frac{\partial C}{\partial \theta_i}\frac{\partial C}{\partial \theta_j} + \frac{n}{C}\frac{\partial^2 C}{\partial \theta_i \partial \theta_j}$$

except for

$$\frac{\partial^2 \log L}{\partial a \partial d} = -\frac{n}{C^2}\frac{\partial C}{\partial a}\frac{\partial C}{\partial d} + \frac{n}{C}\frac{\partial^2 C}{\partial a \partial d} - \sum_{j=1}^{n}\frac{X_j}{1+dX_j},$$

$$\frac{\partial^2 \log L}{\partial b \partial d} = -\frac{n}{C^2}\frac{\partial C}{\partial b}\frac{\partial C}{\partial d} + \frac{n}{C}\frac{\partial^2 C}{\partial b \partial d} + \sum_{j=1}^{n}\frac{X_j}{1+dX_j},$$

$$\frac{\partial^2 \log L}{\partial c^2} = -\frac{n}{C^2}\left(\frac{\partial C}{\partial c}\right)^2 + \frac{n}{C}\frac{\partial^2 C}{\partial c^2} + (\alpha+\beta)\sum_{j=1}^{n}\frac{X_j^2}{\left(1+cX_j\right)^2},$$

$$\frac{\partial^2 \log L}{\partial c^2} = -\frac{n}{C^2}\left(\frac{\partial C}{\partial d}\right)^2 + \frac{n}{C}\frac{\partial^2 C}{\partial d^2} + (a+b)\sum_{j=1}^{n}\frac{X_j^2}{\left(1+dX_j\right)^2},$$

$$\frac{\partial^2 \log L}{\partial c \partial \alpha} = -\frac{n}{C^2}\frac{\partial C}{\partial c}\frac{\partial C}{\partial \alpha} + \frac{n}{C}\frac{\partial^2 C}{\partial c \partial \alpha} - \sum_{j=1}^{n}\frac{X_j}{1+cX_j}$$

and

$$\frac{\partial^2 \log L}{\partial c \partial \beta} = -\frac{n}{C^2} \frac{\partial C}{\partial c} \frac{\partial C}{\partial \beta} + \frac{n}{C} \frac{\partial^2 C}{\partial c \partial \beta} - \sum_{j=1}^{n} \frac{X_j}{1 + cX_j}.$$

Thus, the elements of the FIM are straight-forward upon noting that

$$\begin{aligned} & E\left[\frac{X^m}{(1+cX)^m} \frac{X^n}{(1+dX)^n}\right] \\ & = A(m,n) C c^{1-\alpha-a-m-n} B(m+n+\alpha+a-1, \beta+b+1), \end{aligned}$$

where

$$\begin{aligned} & A(m,n) \\ = \ & {}_2F_1\left(m+n+\alpha+a-1, n+a+b; m+n+\alpha+\beta+a+b; 1-\frac{d}{c}\right). \end{aligned}$$

The FIM in directly useful in statistics to find lower–bounds for variances and co-variances. If $\theta = (a, b, c, d, \alpha, \beta)^T$ and $T(X_1, X_2, \ldots, X_n)$ denotes some statistic of θ then it is well-known that

$$\mathrm{Cov}\left(T\left(X_1, X_2, \ldots, X_n\right)\right) \geq \frac{\partial \phi}{\partial \theta^T} I^{-1}(\theta) \frac{\partial \phi^T}{\partial \theta},$$

where $\phi = E[T(X_1, X_2, \ldots, X_n)]$ and $I^{-1}(\theta)$ denotes the inverse of the FIM, $I(\theta)$, determined above. This inequality is known as the Cramer-Rao Inequality. The exact form of the lower-bound will of course depend on what T is.

7. Conclusions. We have introduced a new generalization of the F distribution motivated by compounding. We have derived various properties of this distribution, including its cdf, moments, mean deviation about the mean, mean deviation about the median, entropy, asymptotic distribution of the extreme order statistics, maximum likelihood estimates and the Fisher information matrix. We expect that this new distribution will prove to be a good model for income data.

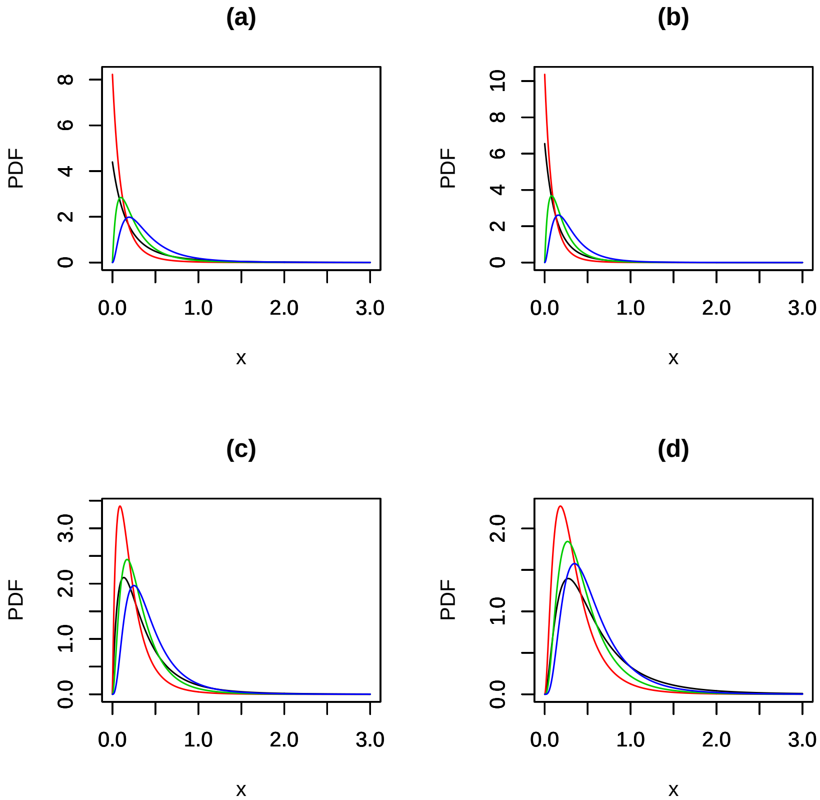

Figure 1. Plots of the pdf of (4) for (a): $(\alpha, \beta) = (1, 1)$; (b): $(\alpha, \beta) = (1, 3)$; (c): $(\alpha, \beta) = (2, 3)$; and, (d): $(\alpha, \beta) = (3, 3)$. The four curves in each plot are: the solid curve $(a = 1, b = 1)$, the curve of lines $(a = 1, b = 3)$, the curve of dots $(a = 2, b = 3)$, and the curve of lines and dots $(a = 3, b = 3)$.

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