ON THE DENSITIES OF SOME SUBSETS OF INTEGERS

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In this note, we prove two conjectures concerning densities of subsets of positive integers suggested in [1] and [2], respectively. Throughout this paper, we use p and q for prime numbers and x for a large positive real number. If $\mathcal{A} \subset \mathbb{N}$ is a subset of the positive integers, we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. We use the Vinogradov symbols $\ll$ and $\gg$, and the Landau symbols O and o with their usual meanings. Namely, we say that $f(x) \ll g(x)$, or that $f(x) = O(g(x))$, if the inequality $|f(x)| < cg(x)$ holds with some positive constant c for all sufficiently large x. The notation $g(x) \gg f(x)$ is equivalent to $f(x) \ll g(x)$, while $f(x) = o(g(x))$ means that $f(x)/g(x)$ tends to zero when x tends to infinity. We use $\log x$ for the natural logarithm of x.

1. Sigma-Primes. Following [1], a positive integer n is called a *sigma-prime* if n and $\sigma(n)$ are coprimes, where $\sigma(n)$ is the sum of the divisors of n. Let $\mathcal{SP}$ be the set of all sigma-primes. It was conjectured in [1] that $\mathcal{SP}$ is of asymptotic density zero. Here, we prove this conjecture.

Theorem 1. The inequality

$$\#\mathcal{SP}(x) \ll \frac{x}{\log\log\log x}$$

holds for all $x > e^e$.

Proof. Let x be a large positive real number. Lemma 4 in [5] asserts that there exists an absolute constant c_1 such that $\sigma(n)$ is divisible by all primes

$$p < y := c_1 \frac{\log\log x}{\log\log\log x}$$

for all $n < x$ except for a subset of such n of cardinality $O(x/\log\log\log x)$. Thus,

$$\#\mathcal{SP}(x) \le \#\{n \le x \ : \ \gcd(n, p) = 1 \ \text{ for all } \ p \le y\} + O\left(\frac{x}{\log\log\log x}\right). \tag{1}$$

On the other hand, by the regular Erathostenes-Legendre sieve (see Theorem 1.1 in [4]), and Mertens's estimate,

$$\#\{n \le x \ : \ \gcd(n,p) = 1 \ \text{ for all } \ p \le y\} \ll x \prod_{p<y} \left(1 - \frac{1}{p}\right) \ll \frac{x}{\log y}$$

$$\ll \frac{x}{\log\log\log x},$$

which together with inequality (1) completes the proof of Theorem 1.

Remark 1. The author of [1] also makes the comment that "the set of prime powers has density zero and that the set of sigma-primes is not much larger". We point out that if we define *phi-primes* in the same way as the *sigma-primes* but with the function $\sigma(n)$ replaced by the Euler function $\phi(n)$, and if we write $\mathcal{PP}$ for the set of all phi-primes, then Erdős [3] showed that the estimate

$$\#\mathcal{PP}(x) = (1 + o(1)) \frac{x e^{-\gamma}}{\log\log\log x},$$

holds as $x \to \infty$, where γ is the Euler constant. Given that the arithmetic properties of the function $\sigma(n)$ resemble the arithmetic properties of the function $\phi(n)$, it is likely that the estimate

$$\#\mathcal{SP}(x) = (1 + o(1)) \frac{c_2 x}{\log\log\log x} \tag{2}$$

holds with some constant c_2 when x tends to infinity. We leave it to the reader to determine whether estimate (2) holds with some constant c_2, and in the affirmative case to compute c_2. If correct, estimate (2) shows that there are "a lot more" sigma-primes than prime powers given that the number of prime powers $p^\alpha \le x$ is only $(1 + o(1))x/\log x$ as x tends to infinity.

2. Ans Numbers. Following [2], a positive integer n is called an *ans number* if it admits a representation of the form $p^2 - q^2$, where p and q are primes. Let $\mathcal{ANS}$ denote the set of all ans numbers. It was conjectured in [2] that $\mathcal{ANS}$ is of asymptotic density zero. Here, we prove this conjecture.

Theorem 2. The inequality

$$\#\mathcal{ANS}(x) \ll \frac{x}{\log x}$$

holds for all $x > 1$.

Proof. Let x be a large positive real number. Let $n < x$ be such that $n = p^2 - q^2 = (p-q)(p+q)$. Write $d = p - q$. Note that $d < p + q$, therefore $d^2 < (p+q)(p-q) = n < x$. Hence, $d < x^{1/2}$. Fix d. Then $2q \le p+q = n/d < x/d$; thus, $q \le x/(2d)$. Hence, in order to get an upper bound on the number of ans numbers $n \le x$ for which d is fixed, it suffices to get an upper bound on the number of primes $q \le x/(2d)$ such that $p = q + d$ is also prime. Let $\mathcal{Q}_d$ denote the set of such primes. The combinatorial sieve (see, for example, Corollary 2.4.1 in [4]), shows that the number of such primes is

$$\#\mathcal{Q}_d \ll \prod_{p|d}\left(1 - \frac{1}{p}\right)\frac{x}{d(\log(x/d))^2} = \frac{x}{\phi(d)(\log(x/d))^2}.$$

Since $d \le x^{1/2}$, we have that $1/(\log(x/d)) \le 2/\log x$, therefore, the above estimate implies

$$\#\mathcal{Q}_d \ll \frac{x}{\phi(d)(\log x)^2}. \tag{3}$$

Summing up inequality (3) over all possible $d \le x^{1/2}$, we get

$$\#\mathcal{ANS}(x) \le \sum_{d \le x^{1/2}} \#\mathcal{Q}_d \ll \frac{x}{(\log x)^2} \sum_{d \le x^{1/2}} \frac{1}{\phi(d)} \ll \frac{x}{\log x},$$

where in the last estimate above we used the well-known fact, due to Landau, that the estimate

$$\sum_{n \le z} \frac{1}{\phi(n)} = c_3 \log z + c_4 + O\left(\frac{\log z}{z}\right) \ll \log z \qquad (z := x^{1/2})$$

holds with some constants c_3 and c_4 for all $z \geq 1$.

References

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