ON A CLASSIFICATION OF LOCALLY 2-SIERPINSKI SPACES

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Abstract. A bijection between topological spaces, locally homeomorphic to the square of Sierpinski space, and graphs without isolated vertices is established. This is used to describe some topological properties of these spaces in combinatorial terms and to disprove a conjecture of Rostami [3] in particular.

1. Introduction. *Locally* 2*-Sierpinski spaces* (lS^2 spaces for short) have been introduced in [3] as extensions of the notion of a locally Sierpinski space given in [1]. A topological space X is said to be an lS^2 space, if every point in X possesses an open neighborhood, U, homeomorphic to the product S^2 of two Sierpinski spaces, i.e., U consists of four points $U = \{a, b_1, b_2, c\}$ and the subspace topology is given by $\{\emptyset, \{a\}, \{a, b_1\}, \{a, b_2\}, U\}$.

As done in [1] and [2] for a locally Sierpinski space the points in X can be partitioned into three pair-wise disjoint sets:

$X_0 = \{x \in X| \; \{x\} \text{ open}\}$, the set of *suns*,

$X_1 = \{x \in X| \; x \text{ has a two point minimal neighborhood}\}$, the set of 1-*satellites*, and

$X_2 = \{x \in X| \; x \text{ has an } S^2\text{-type minimal open neighborhood}\}$, the set of 2-*satellites.*

Let $\mathcal{M}$ be the subset of X, given by

$$\mathcal{M} = \{M \subset X \mid M \text{ is open and homeomorphic to } S^2\}.$$

Since the S^2-type minimal neighborhoods are uniquely associated to their centers, the map

$$\begin{aligned} \Psi : X_2 &\longrightarrow \mathcal{M} \\ x &\to M_x, \end{aligned}$$

* Supported in part by UI&D *Matemática e Aplicações* of University of Aveiro, through Program POCTI of FCT cofinanced by the European Community fund FEDER.

where M_x is the minimal S^2-type neighborhood containing x, is well defined. In fact, Ψ is a bijective map. Besides, the map

$$\begin{aligned} \chi{:}\mathcal{M} &\longrightarrow X_0 \\ M &\to x_0, \end{aligned}$$

where x_0 is the sun of M, is surjective.

Each $M \in \mathcal{M}$ contains exactly two points of X_1. Let $\mathcal{P}_2(X_1)$ denote the set of two-point subsets of X_1 and consider the map,

$$\begin{aligned} \Phi{:}\mathcal{M} &\to \mathcal{P}_2(X_1) \\ M &\to M_2, \end{aligned}$$

where M_2 is the two-point subset of X_1 contained in M.

It can be seen immediately that:

a) For any $M_1, M_2 \in \mathcal{M}$, $\Phi(M_1) \cap \Phi(M_2) \neq \emptyset$ implies $\chi(M_1) = \chi(M_2)$.
b) The connected components of X are given by

$$C_x := \bigcup \{M \in \mathcal{M} | \chi(M) = x\}, \quad x \in X_0,$$

and they are open. Hence, X is the topological sum of its connected components C_x, $x \in X_0$.

Consequently, the essential information for the characterization of the topological structure of X is stored in the interrelations between X_1 and X_2. From now on we shall assume that X is connected, that is, $\# X_0 = 1$.

2. The Associated Graph. Let $(X, \mathcal{T})$ be a connected lS^2 space, $\{*\}$ its unique sun, i.e., $\{*\} = X_0$. We may associate to X a graph, (V, E), as follows:

i) the vertex set, V, is given by X_1, i.e. $V = X_1$;
ii) to every point $x \in X_2$ corresponds exactly one edge $e_x \in E$ connecting the two-point subsets of X_1, which belong to $\Psi(x)$, that is,

$$E = \{e_x \text{ connecting the two 1-satellites in } (\Phi(\Psi(x)) \subset V = X_1\}.$$

It should be pointed out that the same two vertices may be connected by several edges. This graph will be called the *graph of* 1*-satellites* of $(X, \mathcal{T})$.

As will be seen from the proof of the following theorem, this graph contains all information on the lS^2 space X.

Theorem 1. Any graph (V, E) without isolated vertices can be interpreted as the graph of 1-satellites of a connected locally 2-Sierpinski space $(X, \mathcal{T})$. This construction gives a bijective correspondence between these graphs and all homeomorphism classes of connected lS^2 spaces.

Proof. Let $\mathcal{G} = (V, E)$ be a graph without isolated vertices. We may construct an lS^2 space X whose graph of its 1-satellites is precisely $\mathcal{G}$ as follows. Let X be the disjoint union $X_0 \dot{\cup} X_1 \dot{\cup} X_2$, where $X_0 := \{*\}$, $X_1 := V$, and $X_2 := E$. Define,

$$\mathcal{B} := \{*\} \cup \{\{*, v\} | v \in V\} \cup \{\{*, v_1, v_2, e\} |\, e \in E,\ e \text{ connects } v_1 \text{ and } v_2\}.$$

Take set $\mathcal{B}$ for a basis of a topology $\mathcal{T}$ of X. Since sets of type $\{*, v_1, v_2, e\}$, where $e \in E$ connects v_1 and v_2, are open and have the S^2-topology as subspace topology, and (V, E) has no isolated vertices, every point $x \in X$ can be found in one of these sets. This implies that $(X, \mathcal{T})$ is lS^2.

It should be pointed out that the graphs (V, E) under consideration may have several components and cycles with two vertices only, i.e., two distinct edges connecting the same vertices.

The above characterization of lS^2 spaces enables us to establish special types of such spaces using combinatorial terms.

Let $(X, \mathcal{T})$ be a connected lS^2. Then $(X, \mathcal{T})$ is called

- *unsplit*, if the associated graph of 1-satellites (V, E) does not contain any cycle with two vertices only,
- *full*, if $(X, \mathcal{T})$ is unsplit and (V, E) is a complete graph,
- *sparse*, if $(X, \mathcal{T})$ is unsplit and any vertex of (V, E) is an endpoint.

3. Extensions and Reductions of an lS^2. An extension of an lS^2 space $(X, \mathcal{T})$ by *splitting* consists of adding a 2-satellite, which leads to a new cycle with (only) two vertices in the associated graph of 1-satellites. An extension of an lS^2 space $(X, \mathcal{T})$ by *filling* consists in the addition of a 2-satellite, which corresponds to the addition of an edge in the associated graph of 1-satellites connecting vertices which had not been connected before. Extensions of an lS^2 by fillings are sometimes possible, while splittings are always possible. For this situation we have the following proposition.

Proposition 1. An extension of the (connected) lS^2 space $(X, \mathcal{T})$ by splittings or fillings contains the given space as an open dense set.

Proof. According to Theorem 1, such an extension $(\tilde{X}, \tilde{\mathcal{T}})$ will be given by

$$\tilde{X} = \{*\} \dot{\cup} X_1 \dot{\cup} \tilde{X}_2,$$

where

$$X = \{*\} \cup X_1 \cup X_2$$

and $X_2 \subset \tilde{X}_2$.
Taking in account that

$$X = \bigcup \{M \mid M \in \mathcal{M}\} = \bigcup \{M \mid M \in \mathcal{M},\ M \subset X\}$$

and that, by the given construction, any $M \in \mathcal{M}$ is an open subset of $\tilde{X}$, X is an open subset of $\tilde{X}$. Moreover, any neighborhood of $\tilde{x} \in \tilde{X}_2$ will contain $* \in X$ and so X is dense in $\tilde{X}$.

Proposition 2. Every lS^2 space $(X, \mathcal{T})$ can be reduced to an unsplit lS^2 space $(X_u, \mathcal{T}_u)$ such that X_u is an open dense subset of X. Up to a homeomorphism, this subset is uniquely determined.

Proof. Consider the associated graph of 1-satellites $\mathcal{G} = (V, E)$ of $(X, \mathcal{T})$. If $\mathcal{G}$ contains a cycle of 2-vertices, remove one of the edges. This procedure may continue until the reduced graph will have no such cycles. According to Theorem 1, there exists an lS^2 space $(X_u, \mathcal{T}_u)$ associated to the reduced graph. Obviously, $(X, \mathcal{T})$ can be reconstructed, by splitting from $(X_u, \mathcal{T}_u)$ and so, according to Proposition 1, X_u is an open dense subset of X.

In the above procedure, the choice of the edges to be removed is arbitrary. Different choices clearly lead to distinct unsplit lS^2 spaces. Next, we shall show that different choices lead to homeomorphic lS^2 spaces.

Let $\tilde{X}_u$ be a second such reduction. With the obvious notations, we get

$$X_u = \{*\} \dot\cup X_1 \dot\cup X_{u2}, \quad \tilde{X}_u = \{*\} \dot\cup X_1 \cup \tilde{X}_{u2}.$$

Denote by $\Upsilon = \Psi \circ \Phi$. Since, in the unsplit situation, the maps

$$\Upsilon_u : X_{u2} \to \mathcal{P}_2(X_1) \text{ and } \tilde{\Upsilon}_u : \tilde{X}_{u2} \to \mathcal{P}_2(X_1)$$

are injective and have the same image, we can extend the identity on $\{*\} \cup X_1$ by

$$\tilde{\Upsilon}_u^{-1} |_{\Upsilon_u(X_{u2})}) \circ \Phi_u$$

to a bijection F from X_u to $\tilde{X}_u$ which is open and continuous, because 1-point, 2-point, and 4-point open sets are preserved. Hence, F is a homeomorphism.

Remark 1. It follows immediately that

a) $(X,\mathcal{T})$ is unsplit. $\iff \Upsilon\colon X_2 \to \mathcal{P}_2(X_1)$ is injective.
b) Every (not necessarily connected) unsplit lS^2 is a dense subset of a topological sum of full lS^2 spaces.

Remark 2. In [3] M. Rostami conjectured that, in addition to the above conclusions, an lS^2 space X admits a decomposition

$$X_1 = X_1^1 \dot{\cup} X_1^2 \text{ with surjections } p_i : X_2 \to X_1^i, \quad i = 1, 2,$$

such that $\Phi(\Psi(x)) = \{p_1(x), p_2(x)\}$ for all $x \in X_2$.

That this conjecture is not true can be seen as a consequence of Theorem 1. But in the case of a finite X, we can also see this from the following:

a) Assuming that his conjecture holds, it implies that

$$\#X_1 = \#X_1^1 + \#X_1^2 \leq 2\#X_2.$$

Going to the unsplit reduction, we get

$$\begin{aligned} \#\hat{X}_2 = \#\Phi(\Psi(\hat{X}_2)) &\leq \mathcal{P}_2(X_{1)} \\ &\leq \#X_1^1 \cdot \#X_1^2 \\ &\leq \frac{1}{4}(\#X_1)^2. \end{aligned}$$

In terms of the associated graph of 1-satellites of the unsplit reduction, the first inequality will be an equality if every vertex of X_1^1 is connected with every vertex of X_1^2, and the second one will be an equality if $\#X_1^2 = \#X_1^1$. This implies $\#\hat{X}_2 = \frac{1}{4}(\#X_1)^2$. In particular $\#X_1$ has to be even, which reduces the generality proved in Theorem 1. Rostami assumed that the graph has to be bipartite, which does not hold true in general.
b) Rostami's conditions easily can be seen to hold true if the lS^2 is sparse.

The fact that, in lS^2 spaces, each point has a minimal neighborhood, immediately restricts the converging sequences. In fact, if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in an lS^2 space, whose limit is:

a) a sun $*$, then the sequence will stabilize at $\{*\}$;
b) a 1-satellite $x \in X_1$, then the sequence will stabilize in $\{*, x\}$;
c) a 2-satellite $y \in X_2$, then the sequence will stabilize in the S^2-type open subset $\{*, y\} \cup \Upsilon(y)$. In particular, a flipping sequence alternating between the two points in $\Upsilon(y) = \{x_1, x_2\}$ converges to y. That is, the

only case of an alternating sequence between x_1 and x_2 in X_1 that is convergent. Moreover, the limit consists of $\Psi^{-1}(\Phi^{-1}(\{x_1, x_2\}))$.

From these considerations, we get the following proposition.

Proposition 3. An unsplit connected lS^2 space is full if and only if every flipping sequence of 1-satellites is convergent. The limit is a uniquely determined 2-satellite.

A 2-Sierpinski space may be obtained as the orbit space of the operation of a group of homeomorphisms on the unit circle S^1 as follows. Consider S^1 as the Alexandroff compactification of $\mathbb{R}$ and take the operation on $\mathbb{R}$ defined in [2] having three orbits a, b_1, b_2 such that the quotient topology is given by the open sets $\emptyset, \{a\}, \{a, b_1\}, \{a, b_2\}, \{a, b_1, b_2\}$, (which is a Sierpinski space). Keeping ∞ fixed, this operation naturally extends to S^1 leading to the additional orbit $c = \{\infty\}$. Obviously, the only neighborhood of c in the quotient topology is given by $\{a, b_1, b_2, c\}$. It is an open question to determine if lS^2 spaces occur as a quotient of such group operations.

References

1. F. J. Craveiro de Carvalho & A. M. d'Azevedo Breda, "Alguns Passos em Volta do Espaço de Sierpinski," *Boletim da SPM,* 28 (1994), 59–63.

2. F. J. Craveiro de Carvalho & Bernd Wegner, "Locally Sierpinski Spaces as Interval Quotients," *Kyungpook Mathematical Journal,* 42 (2002), 165–170.

3. M. Rostami, "On Locally Sierpinski Spaces," *Kyungpook Mathematical Journal,* 37 (1997), 117–122.

Mathematics Subject Classification (2000): 00A99, 54G99, 05C99

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