A CATALOG OF INTERESTING DIRICHLET SERIES

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Abstract. A Dirichlet series is a series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where the variable s may be complex or real and $f(n)$ is a number-theoretic function. The sum of the series, $F(s)$, is called the generating function of $f(n)$. The Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where n runs through all integers and p runs through all primes is the special case where $f(n) = 1$ identically. It is fundamental to the study of prime numbers and many generating functions are combinations of this function. In this paper, we give an overview of some of the commonly known number-theoretic functions together with their corresponding Dirichlet series.

1. Introduction. There seems to be no convenient list of Dirichlet series except for very large handbooks listing hundreds or thousands of finite and infinite series of all kinds. Even there, many interesting series may be missing and new ones turn up every so often. We believe that a handy catalog of interesting Dirichlet series would be of use to researchers and so we have gone through the literature [1, 2, 3, ..., 43] and compiled such a listing. In particular, the classic texts by Hardy and Wright [20], McCarthy [25], and Titchmarsh [40] proved invaluable in this exercise. Our list has been confined to what we believe to be useful and interesting series. Additions and corrections to this initial catalog are welcomed.

By a Dirichlet series we mean any series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Most of the series we give converge for $\Re(s) > 1$, but we will just list the real interval of convergence. The coefficients $f(n)$ are commonly well-known number-theoretic functions; and for those values of s for which the series converges absolutely, the Dirichlet series serves as a generating function of $f(n)$. Further, if

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \text{ and } G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \text{ for } s > s_0,$$

then

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s}, s > s_0,$$

where

$$(f*g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

is the Dirichlet convolution of f and g. Also, if the Dirichlet inverse f^{-1} of f exists, then

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^s},\ s > s_0.$$

Many of the formulas we list follow from the results above as well as from the fact that when $f(n)$ is a multiplicative function, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \Pi_p \left\{1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + \cdots \right\},$$

where p denotes primes.
The simplest such series is the classical Zeta function when $f(n) = 1$ identically:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

convergent for $s > 1$.
For the sake of completeness we first give a list of some of the standard number-theoretic functions we have found to occur in many Dirichlet series. Let p always designate a prime number, (n, m) the greatest common divisor of natural numbers n and m, and d a divisor of a natural number, as indicated.

2. A Catalog Of Common Number-Theoretic Functions

N-1 $p_n = n$th prime number, [20, page 5–6].

N-2 $\pi(n) = \sum_{p \le n} 1 =$ the number of primes less than or equal to n, [20, page 6].

N-3 $\tau(n) = \sum_{d|n} 1 =$ the number of divisors of n, [20, page 239].

Here we have used the Greek letter tau for 'Teiler' ('divisor' in German). Many British books commonly denote this function by $d(n)$.

N-4 $\tau(n,k)$ = the number of ways of expressing n as a product of k positive factors (of which any number may be unity), expressions in which the order of the factors is different being regarded as distinct.

In particular, $\tau(n,2) = \tau(n)$, [25, page 40].

N-5 $\sigma(n) = \sum_{d|n} d$ = the sum of the divisors of n, [20, page 239].

N-6 $\sigma_k(n) = \sum_{d|n} d^k$ = the sum of the kth powers of the divisors of n. Thus, $\sigma_1(n) = \sigma(n)$, [20, page 239].

N-7 $\phi(n) = \sum_{\substack{1 \le j \le n \\ (j,n)=1}} 1$ = the number of integers less than or equal to n that are relatively prime to n; commonly called Euler's phi function (or totient function), [25, page 1].

N-8 $\mu(n) = \begin{cases} 1; & \text{if } n = 1 \\ 0; & \text{if } p^2 | n \\ (-1)^s; & \text{if } n = p_1 p_2 \cdots p_s \end{cases}$, commonly called the Möbius function, [20, page 234].

N-9 $\mu_k(n) = \begin{cases} 1; & \text{if } n = 1 \\ 0; & \text{if } p^2 | n \\ (-1)^s; & \text{if } n = p_1^k p_2^k \cdots p_s^k \end{cases}$, thus, $\mu_1 = \mu$, [25, page 39].

N-10 $J_k(n)$ = the number of ordered k-tuples $< a_1, a_2, \cdots, a_k >$ where $1 \le a_i \le n$ for $1 \le i \le k$ and $(a_1, a_2, \cdots, a_k, n) = 1$, also known as the Jordan totient function. It is easily shown that $J_k(n) = \sum_{d|n} d^k \mu(\frac{n}{d})$, [25, page 13].

N-11 $\Lambda(n) = \begin{cases} \log p; & \text{if } n = p^\alpha \text{ for } \alpha \ge 1 \\ 0; & \text{otherwise} \end{cases}$, also known as the Von Mangoldt function, [25, page 247].

N-12 $v(n) = \sum_{p|n} 1$ = the number of different prime factors of n. This is sometimes denoted by $\omega(n)$, with $\omega(1) = 0$. The letter 'v' comes from the German word 'verschieden' meaning 'different', [17].

N-13 $\Omega(n) = \sum_{p^i|n} 1$ = the total number of prime factors of n counting repetitions of a prime. Thus, if $n = \Pi_{i=1}^t p_i^{\alpha_i}$, then $\Omega(n) = \sum_{i=1}^t \alpha_i$, [17].

N-14 $\lambda(n) = (-1)^r$, where $r = \Omega(n)$, with $\lambda(1) = 1$, also called Liouville's function, [25, page 45].

N-15 $H_k(n) = \sum_{\substack{1 \le e_i \le n \\ [e_1, e_2, \cdots, e_k] = n}} \phi(e_1)\phi(e_2)\cdots\phi(e_k)$, called the Von Sterneck function, [25, page 14]. Thus, $H_1(n) = \phi(n)$ and it may be shown that $H_k = J_k$.

N-16 $\lfloor x \rfloor$ = the greatest integer $\le x$, and its dual $\lceil x \rceil$ = the least integer $\ge x$, [3, page 72].

N-17 $a(n)$ = the number of non-isomorphic Abelian groups with n elements. $a(n)$ is a multiplicative function studied by P. Erdős and G. Szekeres, [13].

N-18 $\zeta_k(n) = n^k$, where k is a non-negative integer, [25, page 2].

N-19 $\beta(n)$ = the number of integers j such that $1 \le j \le n$ and (j, n) is a square, [25, page 25].

N-20 $\gamma(n) = \begin{cases} 1; & \text{if } n = 1 \\ p_1 p_2 \cdots p_t; & \text{if } n = \Pi_{i=1}^t p_i^{\alpha_i} \end{cases}$, also called the core function, [34].

N-21 $\phi(n, k)$ = the number of integers j such that $1 \le j \le n$ and $(j, n) = (n + k - j, n) = 1$. It follows that, $\phi(n, 0) = \phi(n)$, [25, page 35].

N-22 $\theta(n)$ = the number of ordered pairs $< a, b >$ of positive integers such that $(a, b) = 1$ and $n = ab$. It follows that $\theta(n) = 2^{\omega(n)}$, [25, page 36].

N-23 $\theta_k(n)$ = the number of k-free divisors of n, where k is a positive integer, greater or equal to 2, [25, page 37].

N-24 $\phi(x, n)$ = the number of integers j such that $1 \le j \le x$ and $(j, n) = 1$. It follows that $\phi(n, n) = \phi(n)$, [25, page 38].

N-25 $\Phi_k(n)$ = the number of integers j such that $1 \le j \le n$ and $(j, n)_k = 1$, where $(a, b)_k$ is the largest common kth power divisor of a and b, $k \ge 2$, [25, page 38]. Also known as Klee's function. Clearly, $(a, b)_1 = (a, b)$ and $\Phi_1 = \phi$.

N-26 $\psi_k(n) = \sum_{d|n} d^k |\mu(\frac{n}{d})|$, where $\psi_1 = \psi$ is known as Dedekind's function, [25, page 41].

N-27 $q_k(n) = \begin{cases} |\mu(n^{\frac{1}{k}})|; & \text{if } n \text{ is a } k\text{th power} \\ 0; & \text{otherwise} \end{cases}$, where k is a positive integer, [25, page 41].

N-28 $\beta_k(n)$ = the number of integers j such that $1 \le j \le n^k$ and $(j, n^k)_k$ is a $2k$th power, [25, page 51].

N-29 $\sum_{k=1}^{n}[k,n]$, where $[k,n]$ = l.c.m of k and n, [17].

N-30 $\sum_{k=1}^{n}(k,n)$ and its generalization $\sum_{k=1}^{n} f((k,n))$, where f is any number theoretic function, [17].

N-31 $\chi(n) = \begin{cases} (-1)^{\frac{1}{2}(n-1)} & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$, [25, page 27].

N-32 $R(n)$ = the number of ordered pairs of integers $< a, b >$ such that $n = x^2 + y^2$, [25, page 26]. It follows that, $R_1(n) = \frac{1}{4}R(n) = \sum_{d|n} \chi(d)$.

N-33 $\delta_k(n)$ = the greatest divisor d of n such that $(d,k) = 1$, where k is a positive integer, [25, page 34].

N-34 $\rho_{k,s}(n) = \sum_{\substack{d|n \\ \frac{n}{d} \text{ is an } s\text{th power}}} d^k$, also known as Gegenbauer's function, [25, page 55].

N-35 For integers a and b, let $e(a,b) = e^{\frac{2\pi i a}{b}}$. Now let n be an integer and r a positive integer and define

$$C(n,r) = \sum_{\substack{1 \le x \le r \\ (x,r)=1}} e(nx, r).$$

Also known as Ramanujan's sum, [20, page 247].

N-36 $\nu_s(n) = \begin{cases} 1; & \text{if } n \text{ is an } s\text{th power} \\ 0; & \text{otherwise} \end{cases}$, [25, page 55].

N-37 $\xi_k(n) = \begin{cases} 1; & \text{if } n \text{ is } k\text{-free} \\ 0; & \text{otherwise} \end{cases}$, [25, page 228].

N-38 $\tau_{k,h}(n)$ = the number of ordered k-tuples $< a_1, a_2, \cdots, a_k >$ of positive integers such that each a_i is an hth power and $n = a_1 a_2 \cdots a_k$, [25, page 40].

N-39 $\rho'_{k,t}(n) = \sum_{\substack{d|n \\ d \text{ is a } t\text{th power}}} d^k$, [25, page 232].

N-40 $\gamma'(n) = (-1)^{\omega(n)}\gamma(n)$, the inverse of the core function, [34].

N-41 $\sigma'_k(n) = \sum_{d|n} \lambda(d) d^k$, where k is a non-negative integer, [25, page 235].

N-42 $\phi_r(n) = \sum_{\substack{1\le j\le n \\ (j,n)=1}} j^r$, $r = 0, 1, 2, 3, \ldots$, [17].

3. The Catalog of Dirichlet Series.

D-1
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_p \left(1 - \frac{1}{p^s}\right)^{-1}, \ s > 1,$$

[20, page 245]. More specifically,

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ and } \zeta(2n) = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!},$$

where B_n denotes Bernoulli's number.

D-2 $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$, $s > 1$, [20, page 250].

D-3 $\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta^2(s)$, $s > 1$, [20, page 250].

D-4 $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$, $s > 2$, [20, page 250].

D-5 $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$, $s > 2$, [20, page 250].

D-6 $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k)$, $s > 1$, $s > k$, [20, page 250].

D-7 $\sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta^{'}(s)$, $s > 1$, [20, page 246].

D-8 $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta^{'}(s)}{\zeta(s)}$, $s > 1$, [20, page 253–254].

D-9 $\sum_{n=1}^{\infty} \frac{2^{v(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$, $s > 1$, [20, page 255].

D-10 $\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$, $s > 1$, [20, page 255].

D-11 $\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$, $s > 1$, [20, page 29, 41, 255].

D-12 $\displaystyle\sum_{n=1}^{\infty} \frac{\tau(n,k)}{n^s} = \zeta^k(s)$, $s > 1$, [20, page 255].

D-13 $\displaystyle\sum_{n=1}^{\infty} \frac{\zeta_1(n)}{n^s} = \zeta(s-1)$, $s > 2$, [25, page 189].

D-14 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_{p>2} \frac{1}{1-p^{-s}} = (1-2^{-s})\zeta(s)$, $s > 1$ where we sum over odd n, [25, page 193].

D-15 $\displaystyle\sum_{n=1}^{\infty} \frac{\zeta_k(n)}{n^s} = \zeta(s-k)$, $s > k+1$, [25, page 226].

D-16 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{2}\left(\frac{\zeta(s)}{\zeta(2s)} - \frac{1}{\zeta(s)}\right)$, $s > 1$, where we sum over n such that n is a product of an odd number of primes, [25, page 227].

D-17 $\displaystyle\sum_{n=1}^{\infty} \frac{\omega(n)}{n^s} = \zeta(s) \sum_{p} \frac{1}{p^s}$, $s > 1$, [25, page 227].

D-18 $\displaystyle\sum_{n=1}^{\infty} \frac{\theta(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$, $s > 1$, [20, page 255].

D-19 $\displaystyle\sum_{n=1}^{\infty} \frac{\left(\sum_{k=1}^{n} [k,n]\right)}{n^s} = \frac{1}{2} \frac{\zeta(s-1)}{\zeta(s-2)} \{\zeta(s-2) + \zeta(s-3)\}$, $s > 4$, [18].

D-20 $\displaystyle\sum_{n=1}^{\infty} \frac{\left(\sum_{k=1}^{n} f((k,n))\right)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, $s > 2$, [18].

D-21 $\displaystyle\sum_{n=1}^{\infty} \frac{\lambda(n)\theta(n)}{n^s} = \frac{\zeta(2s)}{\zeta^2(s)}$, $s > 1$, [25, page 227].

D-22 $\displaystyle\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)}$, $s > \frac{1}{k}$, [25, page 228].

D-23 $\displaystyle\sum_{n=1}^{\infty} \frac{\Phi_k(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(ks)}$, $s > 2$, [33].

D-24 $\displaystyle\sum_{n=1}^{\infty} \frac{\xi_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$, $s > 1$, [20, page 255].

D-25 $\displaystyle\sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^s} = \frac{\zeta(s-k)\zeta(2s)}{\zeta(s)}$, $s > k+1$, [25, page 229].

D-26 $\sum_{n=1}^{\infty} \frac{\psi_k(n)}{n^s} = \frac{\zeta(s-k)\zeta(s)}{\zeta(2s)}$, $s > k+1$, [19, 38].

D-27 $\sum_{n=1}^{\infty} \frac{\lambda(n)J_k(n)}{n^s} = \frac{\zeta(s-k)\zeta(s)}{\zeta(2s)}$, $s > k+1$, [25, page 232].

D-28 $\sum_{n=1}^{\infty} \frac{\nu_k(n)}{n^s} = \zeta(ks)$, $s > \frac{1}{k}$, [25, page 228].

D-29 $\sum_{n=1}^{\infty} \frac{\tau_{k,h}(n)}{n^s} = \zeta^k(hs)$, $s > \frac{1}{h}$, [25, page 229].

D-30 $\sum_{n=1}^{\infty} \frac{\rho_{k,t}(n)}{n^s} = \zeta(s-k)\zeta(ts)$, $s > k+1$, [25, page 229].

D-31 $\sum_{n=1}^{\infty} \frac{R(n)}{n^s} = 4\zeta(s)L(s)$, $s > 1$, where $L(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)^s}$, $s > 1$, [20, page 256].

D-32 $\sum_{n=1}^{\infty} \frac{\lambda(n)\rho'_{k,t}(n)}{n^s} = \frac{\zeta(2s)\zeta(2t(s-k))}{\zeta(s)\zeta(t(s-k))}$, $s > k+1$, [25, page 232].

D-33 $\sum_{n=1}^{\infty} \frac{(-1)^{(k+1)\omega(n)}(\gamma')^k(n)J_k(n)}{n^s} = \frac{\zeta(s-k)}{\zeta(s-2k)}$, $s > 2k+1$, [12], chapter X.

D-34 $\sum_{n=1}^{\infty} \frac{\gamma'(n)\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s-2)}$, $s > 3$, [12], chapter X.

D-35 $\sum_{n=1}^{\infty} \frac{\lambda(n)\tau(n,k-1)\tau(n^2\gamma^{k-2}(n))}{(k-1)^{\omega(n)}n^s} = \frac{\zeta^k(2s)}{\zeta^{k+1}(s)}$, $s > 1$, $k \geq 2$, [12], chapter X.

D-36 $\sum_{n=1}^{\infty} \frac{\tau(n,k-2)\tau(n^2\gamma^{k-3}(n))}{(k-2)^{\omega(n)}n^s} = \frac{\zeta^k(s)}{\zeta(2s)}$, $s > 1$, $k \geq 3$, [12], chapter X.

D-37 $\sum_{n=1}^{\infty} \frac{\sigma'_k(n)}{n^s} = \frac{\zeta(s)\zeta(2(s-k))}{\zeta(s-k)}$, $s > k+1$, [25, page 235].

D-38 $\sum_{n=1}^{\infty} \frac{\beta_h(n)\sigma_k(n)}{n^s} = \frac{\zeta(2s)\zeta(s-h)\zeta(s-h-k)\zeta(2(s-k))\zeta(2s-h-k)}{\zeta(s)\zeta(s-k)\zeta(2(2s-h-k))}$, $s > h+k+1$, [25, page 235].

D-39 $\sum_{n=1}^{\infty} \frac{\sigma_h^{'}(n)\sigma_k(n)}{n^s} = \frac{\zeta(s)\zeta(2(s-h))\zeta(s-k)\zeta(2(s-h-k))\zeta(2s-h-k)}{\zeta(s-h)\zeta(s-h-k)\zeta(2(2s-h-k))}$, $s > h+k+1$, [6].

D-40 $\sum_{n=1}^{\infty} \frac{\lambda(n)\sigma_h^{'}(n)\sigma_k(n)}{n^s} = \frac{\zeta(2s)\zeta(2(s-k))\zeta(s-h)\zeta(2s-h-k)}{\zeta(s)\zeta(2(2s-h-k))}$, $s > h+k+1$, [6].

D-41 $\sum_{n=1}^{\infty} \frac{\sigma_h^{'}(n)\sigma_k^{'}(n)}{n^s} = \frac{\zeta(s)\zeta(2(s-k))\zeta(2(s-h))\zeta(s-h-k)}{\zeta(2s-h-k)\zeta(s-h)\zeta(s-k)}$, $s > h+k+1$, [6].

D-42 $\sum_{n=1}^{\infty} \frac{\lambda(n)\sigma_h^{'}(n)\sigma_k^{'}(n)}{n^s} = \frac{\zeta(2s)\zeta(2(s-h-k))\zeta(s-h)\zeta(s-k)}{\zeta(s)\zeta(s-h-k)\zeta(2s-h-k)}$, $s > h+k+1$, [6].

D-43 $\sum_{n=1}^{\infty} \frac{\sigma_k^{'}(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-2k)}{\zeta(s-k)}$, $s > 2k+1$, [25, page 237].

D-44 $\sum_{n=1}^{\infty} \frac{C(kn,r)}{n^s} = \zeta(s)\sum_{d|r} d^{1-s}(k,d)^s\mu(\frac{r}{d})$, $s > 1$, [11].

D-45 $\sum_{n=1}^{\infty} \frac{\lambda(n)C(n,r)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}\sum_{d|r} d^{1-s}\lambda(d)\mu(\frac{r}{d})$, $s > 1$, [11].

D-46 $\sum_{n=1}^{\infty} \frac{\lambda(n)\sigma_h(n)\sigma_k(n)}{n^s} = \frac{\zeta(2s)\zeta(2(s-h))\zeta(2(s-k))\zeta(2(s-h-k))}{\zeta(s)\zeta(s-h)\zeta(s-k)\zeta(s-h-k)\zeta(2s-h-k)}$, $s > h+k+1$, [25, page 232].

D-47 $\sum_{n=1}^{\infty} \frac{\lambda(n)\sigma_k(n)}{n^s} = \frac{\zeta(2s)\zeta(2(s-k))}{\zeta(s)\zeta(s-k)}$, $s > k+1$, [25, page 232].

D-48 $\sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s} = \zeta(s)\Pi_p\left(\frac{p^s+p-1}{p^s}\right)$, $s > 1$, [34, 42].

D-49 $\sum_{n=1}^{\infty} \frac{\lambda(n)\tau(n^2)}{n^s} = \frac{\zeta^2(2s)}{\zeta^3(s)}$, $s > 1$, [25, page 234].

D-50 $\sum_{n=1}^{\infty} \frac{\lambda(n)\tau^2(n)}{n^s} = \frac{\zeta^3(2s)}{\zeta^4(s)}$, $s > 1$, [25, page 234].

D-51 $\sum_{n=1}^{\infty} \frac{\sigma_k(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-k)\zeta(s-2k)}{\zeta(s(s-k))}$, $s > 2k+1$, [25, page 237].

D-52 $\sum_{n=1}^{\infty} \frac{J_k(n)}{n^s} = \frac{\zeta(s-k)}{\zeta(s)}$, $s > k+1$, [25, page 226].

D-53 $\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$, $s > 1$, [20, page 255].

D-54 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$, $s > 1$, [40, page 21].

D-55 $\sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)}$, $s > 1$, [40, page 5].

D-56 $\sum_{n=1}^{\infty} \frac{\tau^2(n,k)}{n^s} = \zeta^k(s)\Pi_p P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right)$, $s > 1$, where $P_k(x)$ is the kth Legendre polynomial in x, [5, Vol. 3, page 170].

D-57 $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n}\frac{1}{n^s} = \log(\zeta(s))$, $s > 1$, [40, page 57].

D-58 $\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$, for $s > 1$, $s-a > 1$, $s-b > 1$, and $s-a-b > 1$, [20 page 29, 43, 56].

D-59 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} = \frac{1-2^{2k-1}}{(2k)!}|B_{2k}|$, $k = 1, 2, 3, \ldots$, where $B_k(x)$ is the Bernoulli polynomial of degree k in x, [3, pages 266-267].

D-60 $\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1}\pi^{2k}}{(2k)!}|B_{2k}| = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}$, $k = 1, 2, 3, \ldots$, with $B_n = B_n(0)$, [3, pages 266–267].

D-61 $\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} = (-1)^{k-1}\frac{(2\pi)^{2k+1}}{2(2k+1)!}\int_0^1 B_{2k}(x)\cot \pi x dx$, [3, pages 266–267].

D-62 $\sum_{n=1}^{\infty} \frac{f_k(n)}{n^s} = (\zeta(s)-1)^k$, $s > 1$, where $f_k(n)$ is the number of representations of n as a product of k factors, each greater than 1 when $n > 1$, the order of the factors being essential, [40, page 7].

D-63 $\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{1}{2-\zeta(s)}$, where $f(n)$ is the number of representations of n as a product of factors greater than unity, representations with factors in a different order being regarded as distinct, and $f(1) = 1$. It must be noted that $\zeta(s) = 2$ for some $s = \alpha$ a real number greater than 1, so that the result is valid for $s > \alpha$, [40, page 7].

D-64 $\sum_{n=1}^{\infty} \frac{G_n(a,b)}{n^s} = \zeta(s)\zeta(s+a+b) - \zeta(s+a)\zeta(s+b)$, provided $s > 1$, $s+a > 1$, $s+b > 1$ and $s+a+b > 1$, where $G_n(a,b) = \sum_{\substack{d|n \\ d<\sqrt{n}}} \left(d^a - (\frac{n}{d})^a\right)\left(d^b - (\frac{n}{d})^b\right)$, [16].

D-65 $\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \Pi_{n=1}^{\infty}\zeta(ns)$, $s > 1$, [13].

D-66 $\sum_{p} \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns)$, [40, page 12].

D-67 $\sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns)$, [40, page 12].

D-68 $\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \zeta(s)$, where $b(n)$ is the number of divisors of n which are primes or powers of primes, [40, page 12].

D-69 $\sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \zeta(s-1)\frac{1-2^{1-s}}{1-2^{-s}}$, $s > 2$, where $c(n)$ is the greatest odd divisor of n, [40, page 6].

D-70 $\sum_{n=1}^{\infty} \frac{\phi_r(n)}{n^s} = 1 + \frac{1}{(r+1)\zeta(s-r)} \sum_{k=1}^{r+1} \binom{r+1}{k} B_{r+1-k}\zeta(s-k)$, $s > k+1$, $s > r+1$, [17].

D-71

$$\sum_{n=1}^{\infty} \frac{\sum_{j=1}^{n} [j,n]^k}{n^s}$$
$$= \zeta(s-k)\left\{1 + \frac{1}{\zeta(s-2k)(k+1)} \sum_{i=1}^{k+1} \binom{k+1}{i} B_{k+1-i}\zeta(s-k-i)\right\},$$

$s > 2k+1$, $k \geq 1$, [17].

We include the following curious formulas which are special cases of the function defined by $P(x,s) = \sum_{n=1}^{\infty} A_n \frac{x^n}{n^s}$, which is both a power series and a Dirichlet series.

The following five infinite series may be found in the classic text by Whittaker and Watson, chapter seven, [41]. For further references, the reader may also consult W. Spence [35] and L. J. Rogers [32].

D-72 $\lim_{x\to 1}(1-x)^{1-s}\sum_{n=1}^{\infty}\frac{x^n}{n^s}=\Gamma(1-s).$

D-73 Let $\Phi(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^3}$. Then, for $|x|\le\frac{1}{2}$, this function satisfies the functional equation,

$$\Phi\left(\frac{x}{x-1}\right)+\Phi(x)+\Phi(1-x)-\Phi(1)$$
$$=\frac{\pi^2}{6}\log(1-x)+\frac{1}{6}\left\{\log(1-x)\right\}^2\left\{\log(1-x)-3\log x\right\}.$$

D-74

$$\sum_{n=1}^{\infty}\frac{\left(\frac{3-\sqrt{5}}{2}\right)^n}{n^3}$$
$$=\frac{2}{15}\left[6\zeta(3)+\pi^2\log\left(\frac{-1+\sqrt{5}}{2}\right)-5\left\{\log\left(\frac{-1+\sqrt{5}}{2}\right)\right\}^3\right]$$
$$=0.4023504\ldots \text{ approximately.}$$

D-75 $\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^3}=\frac{x^3-\pi^2x}{12}.$

D-76 Let $H(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. This function satisfies the (Abel) functional equation,

$$H(x)+H(y)+H(xy)+H\left(\frac{x(1-y)}{1-xy}\right)+H\left(\frac{y(1-x)}{1-xy}\right)=3H(1).$$

If we define $L(x)=H(x)+\frac{1}{2}\log x\times\log(1-x)$, then Leonard J. Rogers [1906/07], [32] found the following remarkable relations:

$$L(x)+L(1-x)=L(1),$$
$$L(x)+L(y)=L(xy)+L\left(\frac{x(1-y)}{1-xy}\right)+L\left(\frac{y(1-x)}{1-xy}\right),$$
$$L(x)+L(x^2)=L(1),L\left(\frac{\sqrt{5}-1}{2}\right)=\frac{3}{5}L(1)=\frac{\pi^2}{10},$$
$$L\left(\frac{3-\sqrt{5}}{2}\right)=\frac{2}{5}L(1)=\frac{\pi^2}{15}.$$

5. Some General Theorems.

D-77 $\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \Pi_{p\not| r}\left\{1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\frac{f(p^3)}{p^{3s}}+\cdots\right\}$, where r is any positive integer, $(n,r)=1$ and $f(n)$ is any multiplicative function, [25, page 192].

D-78 Let $\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = F(s)$, $s > s_0$, say, where f is any arithmetic function. Then for a positive integer r,

$$\sum_{n=1}^{\infty}\frac{f(n)C(n,r)}{n^s} = \sum_{d|r}\left(d^{1-s}\mu(\frac{r}{d})\sum_{m=1}^{\infty}\frac{f(md)}{m^s}\right),$$

$s > s_0$, [25, page 240].

D-79 $\zeta(s)\sum_{n=1}^{\infty}\frac{a_n}{n^s} = \sum_{n=1}^{\infty}\frac{A_n}{n^s} \Rightarrow \sum_{n=1}^{\infty}a_n\frac{x^n}{1-x^n} = \sum_{n=1}^{\infty}A_n x^n$, [20, page 257].

D-80

$$\zeta(s)(1-2^{1-s})\sum_{n=1}^{\infty}\frac{a_n}{n^s} = \sum_{n=1}^{\infty}\frac{B_n}{n^s}$$
$$\Rightarrow \sum_{n=1}^{\infty}a_n\frac{x^n}{1+x^n} = \sum_{n=1}^{\infty}B_n x^n,$$

[20, pages 247, 257].

D-81 $\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{\pi^2}{6}\sum_{k=0}^{\infty}(-1)^k\frac{2k+5}{(2k+3)!}B_{2k}\pi^{2k}$, posed by W. Jurkat [21]. It is a special case of:

D-82 $\zeta(2n+1) = \zeta(2n)\left\{\frac{1}{2}A_0^{(n)} - \sum_{k=1}^{\infty}A_{2k}^{(n)}\zeta(2k)\right\}$, where the $A_{2k}^{(n)}$ are rational coefficients. This was published by Miklás Mikolás [27], 1953.

D-83 Let $F(s) = \sum_{n=1}^{\infty}\frac{A_n}{n^s}$ and $[f(s)]^p = \sum_{n=1}^{\infty}\frac{A_n(p)}{n^s}$. Then

$$\sum_{d|n}(p+q\log d)A_d(a)A_{\frac{n}{d}}(c) = \frac{p(a+c)+qa\log n}{a+c}A_n(a+c).$$

This was proved in Gould [15]. As shown there, this is a dual of a corresponding identity for powers of power series.

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