ALTERNATIVE FORMULA FOR JACOBI POLYNOMIALS

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Abstract. An alternative (hitherto unknown) representation is derived for Jacobi polynomials.

1. Introduction. Jacobi polynomials are some of the most fundamental tools in algebra. They have applications in almost every branch of mathematics. The usual definition is given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \quad (1)$$

where $P_n(\cdot)$ denotes the Jacobi polynomial of order n. Several representations are available [1] in the mathematics literature for computing (1). In this note, we derive an alternative formula for (1) based on the concept of expectations. To the best of our knowledge, this formula does not appear to have been noticed before.

2. Main Result.

Theorem. If X_1 and X_2 are independent gamma random variables specified by the probability density functions

$$f_{X_1}(x_1) = \frac{x_1^{-\alpha-1}\exp(-x_1)}{\Gamma(-\alpha)}$$

and

$$f_{X_2}(x_2) = \frac{x_2^{-\beta-1}\exp(-x_2)}{\Gamma(-\beta)},$$

respectively, then

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\sum_{l=n-k}^{n}\binom{n}{k}\binom{n}{l}\frac{(-1)^l(1-x)^{n-k}(1+x)^{n-l}}{2^n(n-k-l)!}E\left[\left(\frac{X_2}{1+x}-\frac{X_1}{1-x}\right)^{n-k-l}\right]$$

for $n \geq 1$.

Proof. Using the definition in (1), one can write

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]\\
&= \frac{1}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dt^n}\left[(1-x+t)^{\alpha+n}(1+x-t)^{\beta+n}\right]\\
&= \frac{1}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\\
&\quad\times\frac{d^n}{dt^n}\left[(1-x+t)^n(1+x-t)^n(1-x+t)^{\alpha}(1+x-t)^{\beta}\right]\\
&= \frac{1}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\\
&\quad\times\frac{d^n}{dt^n}\Bigg[\left\{\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}\binom{n}{l}(1-x)^{n-k}(1+x)^{n-l}(-1)^l t^{k+l}\right\}\\
&\quad\times(1-x+t)^{\alpha}(1+x-t)^{\beta}\Bigg]\\
&= \frac{1}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\\
&\quad\times\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}\binom{n}{l}(1-x)^{n-k}(1+x)^{n-l}(-1)^l\\
&\quad\times\frac{d^n}{dt^n}\left[t^{k+l}(1-x+t)^{\alpha}(1+x-t)^{\beta}\right]\\
&= \frac{1}{2^n n!}\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}\binom{n}{l}(1-x)^{n-k}(1+x)^{n-l}(-1)^l\\
&\quad\times\frac{d^n}{dt^n}\left[t^{k+l}\left(1+\frac{t}{1-x}\right)^{\alpha}\left(1-\frac{t}{1+x}\right)^{\beta}\right]\\
&= \frac{1}{2^n n!}\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}\binom{n}{l}(1-x)^{n-k}(1+x)^{n-l}(-1)^l\\
&\quad\times\frac{d^n}{dt^n}\left[t^{k+l}E\left\{\exp\left(-\frac{tX_1}{1-x}\right)\right\}E\left\{\exp\left(\frac{tX_2}{1+x}\right)\right\}\right]\\
&= \frac{1}{2^n n!}\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}\binom{n}{l}(1-x)^{n-k}(1+x)^{n-l}(-1)^l\\
&\quad\times\frac{d^n}{dt^n}\left[t^{k+l}E\left[\exp\left\{t\left(\frac{X_2}{1+x}-\frac{X_1}{1-x}\right)\right\}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2^n n!} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k} \binom{n}{l} (1-x)^{n-k} (1+x)^{n-l} (-1)^l \\
&\quad \times \frac{d^n}{dt^n} \left[t^{k+l} E \left[\sum_{m=0}^{\infty} \left\{ \left(\frac{X_2}{1+x} - \frac{X_1}{1-x} \right)^m \frac{t^m}{m!} \right\} \right] \right] \\
&= \frac{1}{2^n n!} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k} \binom{n}{l} (1-x)^{n-k} (1+x)^{n-l} (-1)^l \\
&\quad \times \frac{d^n}{dt^n} \left[t^{k+l} \sum_{m=0}^{\infty} \left[E \left\{ \left(\frac{X_2}{1+x} - \frac{X_1}{1-x} \right)^m \right\} \frac{t^m}{m!} \right] \right],
\end{aligned}
$$

where we have used the facts $E[\exp(sX_1)] = (1-s)^{\alpha}$ and $E[\exp(sX_2)] = (1-s)^{\beta}$. The result of the theorem follows.

Reference

1. S. Iyanaga and Y. Kawada, Y. (eds), "Jacobi Polynomials, Appendix A, Table 20.V," p. 1480 in *Encyclopedic Dictionary of Mathematics*, MIT Press, Cambridge, MA, 1980.

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