

IDENTIFICATION OF SOME SPATIALLY VARIABLE PHYSICAL PROPERTIES

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Abstract. An inverse problem for the identification of an unknown spatially dependent coefficient in a parabolic partial differential equation is considered as an application for this new technique. An integral identity which explicitly relates changes in coefficients to changes in measured data is presented. Using this identity, it is possible to show that the coefficient to data map is continuous, strictly monotone and injective. Applying a modified Backus-Gilbert method to this identity generates a sequence of coefficients converging to the required unknown coefficient. Finally, implementation of the procedure is discussed and some numerical experiments are displayed.

1. Introduction. Some inverse problems are generated by the need to determine various internal physical properties from external measurements [7]. The identification of a diffusion coefficient is chosen here as a prototype coefficient identification problem that has been approached by various methods. The most common technique for identifying an unknown coefficient from some measured output is the method of least squares [9]. The method described in this paper is based on an idea that goes back to Backus and Gilbert [2]. Mass and Louis [11] provided a mathematical context for the Backus-Gilbert idea. A particular feature of this paper is to apply the Backus-Gilbert method for an integral identity relating changes in an unknown coefficient to corresponding changes in measured data in order to generate a sequence of “trial coefficients” that converges to the required unknown coefficient. The results of a few numerical experiments are provided here to illustrate the working of the method.

2. Derivation of the Integral Identity. Consider the parabolic partial differential equation,

$$\begin{aligned} \partial_t u(x, t) &= \partial_x [K(x) \partial_x u(x, t)] & 0 < x < 1, \quad 0 < t < T \\ u(x, 0) &= 0 & 0 < x < 1 \\ \partial_x u(0, t) &= 0 & 0 < t < T \\ u(1, t) &= f(t) = t & 0 < t < T. \end{aligned} \tag{2.1}$$

We will suppose in all of what follows that the input data f and the coefficient K satisfy,

- (i) $f \in \mathbb{C}^1 [0, T]$ with $f(0) = 0$ and $f'(t) > 0$ for $0 < t < T$.

- (ii) $a \leq K(x) \leq b$, and $|K(x) - K(y)| \leq c|x - y|$ for positive constants a, b, c .

We shall say that such functions f and K are *admissible*. When functions f and K are admissible, the so called direct problem (2.1) has a unique weak solution [13] $u = u(x, t; K)$ belonging to $H^0[0, T : H^1(0, 1)] \cap C[0, T : H^0(0, 1)]$, where $H^s(0, 1)$ is the Sobolev space of order s , $s \geq 0$ [1].

For the direct problem (2.1), we consider the inverse problem in which the coefficient, $K(x)$, is to be identified from a single measurement of output, $K(1) \partial_x u(1, t; K)$, (i.e., the flux at $x = 1$):

$$g(t) = K(1) \partial_x u(1, t), \quad 0 < t < T. \quad (2.2)$$

Alternatively, the identification of $K(x)$ could be based on the measurement of $u(0, t; K)$ or even on simultaneous measurement of both quantities. Additionally, boundary conditions other than those included in (2.1) may be considered with corresponding alternative choices of measured output in place of (2.2). For purposes of this presentation, we will consider the single measurement (2.2).

We collect all necessary results in the next lemma, see also [3], [4] and [6], where similar proofs for slightly different problems are to be found.

Lemma 2.1. For admissible coefficients K_1 and K_2 , let $g_1(t)$ and $g_2(t)$ denote the corresponding measured outputs (2.2). Let $\phi = \phi(x, t)$ denote the solution of the following adjoint problem to (2.1):

$$\begin{aligned} \partial_t \phi(x, t) + \partial_x [K_1(x) \partial_x \phi(x, t)] &= 0 & 0 < x < 1, \quad 0 < t < T \\ \phi(x, T) &= 0 & 0 < x < 1 \\ \partial_x \phi(0, t) &= 0 & 0 < t < T \\ \phi(1, t) &= \theta(t) = T - t & 0 < t < T. \end{aligned} \quad (2.3)$$

Then

(i)

$$\begin{aligned} & \int_0^T [g_1(t) - g_2(t)] \phi(1, t) dt \\ &= \int_0^T \int_0^1 [K_1(x) - K_2(x)] \partial_x u(x, t, K_2) \partial_x \phi(x, t) dx dt. \end{aligned}$$

- (ii) $g_i(t) > 0$ for $t > 0$, $i = 1, 2$.
 (iii)

$$\partial_x u(x, t, K_i) > 0 \quad \text{a.e. on } Q_T = (0, 1) \times (0, T), \quad i = 1, 2. \quad (2.4)$$

- (iv) If $\theta(T) = 0$ and $\partial_t \theta < 0$ for $0 < t < T$, then $\partial_x \phi(x, t) > 0$ a.e. on Q_T .
 (v) If $K_1(x) > K_2(x)$ for $0 < x < 1$, then $g_1(t) > g_2(t)$ for $0 < t < T$.

Result (i) is the previously mentioned integral identity relating changes in the coefficient to the corresponding change in the measured output. Result (v) implies that the coefficient to data mapping is monotone.

Proof. Let $\psi(x, t)$ be an arbitrary test function. To prove (i), write

$$\begin{aligned} & \int \int_{Q_T} [\partial_t u(x, t, K_1) - \partial_t u(x, t, K_2)] \psi(x, t) dx dt \\ &= \int \int_{Q_T} \partial_x [K_1(x) \partial_x u(x, t, K_1)] \psi(x, t) dx dt \\ & \quad - \int \int_{Q_T} \partial_x [K_2(x) \partial_x u(x, t, K_2)] \psi(x, t) dx dt. \end{aligned}$$

Integrating by parts and requiring $\psi(x, t)$ to satisfy (2.3) (i.e., choose $\psi = \phi$) shows that (2.4) must hold.

To prove (ii), note first that the strong maximum principle for parabolic equations implies

$$u(x, 0) = f(0) < u(x, t) < f(T) = u(1, T) \text{ for } 0 < x < 1 \text{ and } 0 < t < T.$$

In particular,

$$u(1 - \varepsilon, t) < u(1, t) \text{ for } 0 < x < 1.$$

Then

$$\frac{u(1, t) - u(1 - \varepsilon, t)}{\varepsilon} > 0 \text{ for } 0 < t < T.$$

It follows that $\partial_x u(1, t) > 0$ for $0 < t < T$ which means that $g(t) > 0$, since $K(1) > 0$. Note also that since $u(x, t, K)$ is the solution to a linear, parabolic problem; the function $g(t) = K(1) \partial_x u(1, t)$ is very smooth on $(0, T)$. In particular, pathological behavior like infinitely rapid oscillation is precluded for such functions.

To prove (iii), let $\psi(x, t)$ be an arbitrary test function and write

$$\int \int_{Q_T} (\partial_t u(x, t, K) - \partial_x [K(x) \partial_x u(x, t, K)]) \partial_x \psi(x, t) dx dt = 0.$$

Integration by parts reduces this to

$$\begin{aligned} & \int \int_{Q_T} \partial_x u [\partial_t \psi(x, t) + K(x) \partial_{xx} \psi(x, t)] dx dt \tag{2.5} \\ &= \int_0^T [K(1) \partial_x u(1, t) \partial_x \psi(1, t) + u(1, t) \partial_t \psi(1, t) - u(0, t) \partial_t \psi(0, t)] dt \\ &+ \int_0^1 [u(x, T) \partial_x \psi(x, T) - u(x, 0) \partial_x \psi(x, 0)] dx. \end{aligned}$$

Now choose ψ to be the solution of

$$\begin{aligned} \partial_t \psi(x, t) + K(x) \partial_{xx} \psi(x, t) &= F(x, t) & 0 < x < 1, \quad 0 < t < T \\ \psi(x, T) &= 0 & 0 < x < 1 \\ \psi(0, t) &= 0 & 0 < t < T \\ \psi(1, t) &= 0 & 0 < t < T. \end{aligned}$$

Then $\partial_x \psi(x, T) = \partial_t \psi(1, t) = \partial_t \psi(0, t) = 0$, and this, together with (2.2) simplifies (2.5) to

$$\int \int_{Q_T} \partial_x u(x, t) F(x, t) dx dt = \int_0^T g(t) \partial_x \psi(1, t) dt. \quad (2.6)$$

The maximum principle applied to the adjoint problem for $\psi(x, t)$ shows that for an arbitrary continuous and nonnegative function $F(x, t)$ we have $\psi(x, t) \leq 0$ on Q_T . Then $\psi(1, t) = 0$ implies that $\partial_x \psi(1, t) > 0$. This fact, together with (2.4 ii) shows that the right side of (2.6) is nonnegative for every continuous and nonnegative function $F(x, t)$, which is to say, $\partial_x u(x, t) \geq 0$ on Q_T in the sense of distributions. The regularity of the solution $u(x, t)$ for (2.1) then implies that $\partial_x u(x, t) \geq 0$ a.e. on Q_T .

To prove (2.4 iv), an argument similar to that used in the proof of (2.4 iii), applied to the adjoint problem (2.3), shows that the assumption on $\theta(t)$ implies $\partial_x \phi(x, t) > 0$ a.e. on Q_T .

Finally, to prove (v), note first that $g_1(t) < g_2(t)$ for $0 < t < T$ leads to an immediate contradiction with (i), (iii), and (iv) of (2.4). Now suppose that for some τ such that $0 < \tau < T$, we have $g_1(t) > g_2(t)$ for $0 < t < \tau$, and $g_1(t) < g_2(t)$ for $\tau < t < T$. Then, applying (2.4 i), first on $(0, 1) \times (0, \tau)$ and then on $(0, 1) \times (0, T)$, leads to

$$\begin{aligned} & \int_0^\tau [g_1(t) - g_2(t)] \phi(1, t) dt \\ &= \int_0^\tau \int_0^1 [K_1(x) - K_2(x)] \partial_x u(x, t, K_2) \partial_x \phi(x, t) dx dt \end{aligned}$$

and

$$\begin{aligned} & \int_0^T [g_1(t) - g_2(t)] \phi(1, t) dt \\ &= \int_0^T \int_0^1 [K_1(x) - K_2(x)] \partial_x u(x, t, K_2) \partial_x \phi(x, t) dx dt. \end{aligned}$$

Then, it follows by subtracting that

$$\begin{aligned} & \int_\tau^T [g_1(t) - g_2(t)] \phi(1, t) dt \\ &= \int_\tau^T \int_0^1 [K_1(x) - K_2(x)] \partial_x u(x, t, K_2) \partial_x \phi(x, t) dx dt. \end{aligned}$$

However, (iii) and (iv) of (2.4) together with the hypothesis that $K_1(x) > K_2(x)$ for $0 < x < 1$, imply that the right side of this last expression is positive, while $g_1(t) < g_2(t)$ for $\tau < t < T$ implies that the left side is negative. As remarked in the proof of (ii), pathological behavior for $g(t)$ is precluded and it follows that when $K_1(x) > K_2(x)$ for $0 < x < 1$, anything other than $g_1(t) > g_2(t)$ for $0 < t < T$ leads to a similar contradiction of (2.4 i).

The results in this lemma imply the injectivity of the coefficient to data mapping $K \rightarrow g$.

For a given direct problem, the integral identity (2.4 i) asserts that if the coefficient K is changed from K_1 to K_2 , there will be a corresponding change in the output $g(t)$ from g_1 to g_2 . On the other hand, if g_1 , corresponding to an unknown coefficient K_1 , is measured experimentally and if, for a given coefficient K_2 , the output g_2 is obtained by solving (2.1), then the integral identity provides an integral equation, which may be solved for the unknown $K_1(x)$. In the next sections, the Backus-Gilbert method is applied to the integral equation to approximate the unknown coefficient $K(x)$.

3. Approximate Solution to the Integral Equation. We introduce the Backus-Gilbert method in (3.1), and in (3.2) we apply this method to the solution of integral equation arising out of the integral identity (2.4 i).

3.1 The Backus-Gilbert Method. Consider the problem of approximating an unknown function $\kappa(x)$, from a finite set of moments $\mu_i = \langle \kappa, \sigma_i \rangle_{H^s \times H^{-s}}$, where $\sigma_i \in H^{-s}(0, 1)$ are known (generalized) functions, $i = 1, 2, \dots, N$. This problem can be stated as follows.

Define $\Lambda: H^s(0, 1) \rightarrow \mathbb{R}^N$,

$$\kappa \rightarrow \Lambda(\kappa) = (\langle \kappa, \sigma_1 \rangle_{H^s \times H^{-s}}, \dots, \langle \kappa, \sigma_N \rangle_{H^s \times H^{-s}}).$$

Given $(\sigma_1, \dots, \sigma_N)$ $\sigma_i \in H^{-s}(0, 1)$ and $(\mu_1, \dots, \mu_N) \in \mathbb{R}^N$.

Find $\kappa \in H^s(0, 1)$ such that $\Lambda(\kappa) = (\mu_1, \dots, \mu_N)$.

Now, for $x_0 \in (0, 1)$, fixed, assume that

$$\kappa(x_0) = \sum_{j=1}^N \Phi_j(x_0) \mu_j. \quad (3.1)$$

Here, the quantities $\mu_j = \langle \kappa, \sigma_j \rangle_{H^s \times H^{-s}}$ are given and $\Phi_j(x_0)$ are unknown functions to be determined. Then

$$\kappa(x_0) = \sum_{j=1}^N \Phi_j(x_0) \langle \kappa, \sigma_j \rangle_{H^s \times H^{-s}} = \left\langle \kappa, \sum_{j=1}^N \Phi_j(x_0) \sigma_j \right\rangle_{H^s \times H^{-s}}.$$

But this implies

$$\sum_{j=1}^N \Phi_j(x_0) \sigma_j(x) = \delta(x - x_0), \quad (3.2)$$

where $\delta(x - x_0)$ is the Dirac distribution concentrated at x_0 [1].

Since $\langle \Lambda(\kappa), \vec{v} \rangle_{\mathbb{R}^N} = \langle \kappa, \Lambda^\top(\vec{v}) \rangle_{H^s \times H^{-s}}$, where Λ^\top denotes the transpose of the operator Λ , and

$$\begin{aligned} \langle \Lambda(\kappa), \vec{v} \rangle_{\mathbb{R}^N} &= (\langle \kappa, \sigma_1 \rangle_{H^s \times H^{-s}}, \dots, \langle \kappa, \sigma_N \rangle_{H^s \times H^{-s}}) \cdot \vec{v} \\ &= \sum_{j=1}^N \langle \kappa, \sigma_j \rangle_{H^s \times H^{-s}} v_j = \left\langle \kappa, \sum_{j=1}^N v_j \sigma_j \right\rangle_{H^s \times H^{-s}}, \end{aligned}$$

it follows that

$$\Lambda^\top: \mathbb{R}^N \rightarrow H^{-s}(0, 1), \quad \Lambda^\top(\vec{v}) = \sum_{j=1}^N v_j \sigma_j(x).$$

Then equation (3.2) is equivalent to

$$\Lambda^\top(\vec{\Phi}(x_0)) = \sum_{j=1}^N \Phi_j(x_0) \sigma_j(x) = \delta(x - x_0). \quad (3.3)$$

To solve equation (3.3) for $\vec{\Phi}(x_0)$, we have to acknowledge that, in general, there will be no solution. However, the normal equation [7]

$$(\Lambda^\top)^* \Lambda^\top(\vec{\Phi}(x_0)) = (\Lambda^\top)^* \delta(x - x_0)$$

is always uniquely solvable, where $(\Lambda^\top)^*$ is the adjoint of Λ^\top and is defined by

$$(\Lambda^\top)^*: H^{-s}(0, 1) \rightarrow \mathbb{R}^N, \quad \langle \Lambda^\top(\vec{v}), G \rangle_{H^{-s}} = \vec{v} \cdot (\Lambda^\top)^*(G).$$

Note that

$$\langle \Lambda^\top(\vec{v}), G \rangle_{H^{-s}} = \left\langle \sum_{j=1}^N v_j \sigma_j(x), G \right\rangle_{H^{-s}} = \sum_{j=1}^N v_j \langle \sigma_j, G \rangle_{H^{-s}},$$

which implies that $(\Lambda^\top)^*(G) = (\langle \sigma_1, G \rangle_{H^{-s}}, \dots, \langle \sigma_N, G \rangle_{H^{-s}})$.

Now,

$$\begin{aligned}
(\Lambda^\top)^* \Lambda^\top \left(\vec{\Phi}(x_0) \right) &= (\Lambda^\top)^* \left[\sum_{j=1}^N \Phi_j(x_0) \sigma_j(x) \right] \\
&= \sum_{j=1}^N \Phi_j(x_0) (\Lambda^\top)^* [\sigma_j(x)] \\
&= \sum_{j=1}^N \Phi_j(x_0) \left(\langle \sigma_1, \sigma_j(x) \rangle_{H^{-s}}, \dots, \langle \sigma_N, \sigma_j(x) \rangle_{H^{-s}} \right) \\
&= \sum_{j=1}^N \Phi_j(x_0) \left(\langle J\sigma_1, \sigma_j(x) \rangle_{H^s \times H^{-s}}, \dots, \langle J\sigma_N, \sigma_j(x) \rangle_{H^s \times H^{-s}} \right)
\end{aligned}$$

and

$$\begin{aligned}
(\Lambda^\top)^* \delta(x - x_0) &= \left(\langle \sigma_1, \delta_{x_0} \rangle_{H^{-s}}, \dots, \langle \sigma_N, \delta_{x_0} \rangle_{H^{-s}} \right) \\
&= \left(\langle J\sigma_1, \delta_{x_0} \rangle_{H^s \times H^{-s}}, \dots, \langle J\sigma_N, \delta_{x_0} \rangle_{H^s \times H^{-s}} \right),
\end{aligned}$$

where $J: H^{-s}(0, 1) \rightarrow H^s(0, 1)$ denotes the duality isomorphism [12] defined by

$$J\sigma(x) = \sum_{n=1}^{\infty} \lambda_n^{-s} \langle \sigma, \omega_n \rangle_{H^s \times H^{-s}} \omega_n(x)$$

and where $\{\omega_n\}$ denotes any suitable orthonormal basis of eigenfunctions in $H^s(0, 1)$ with positive eigenvalues $\{\lambda_n^2\}$. In this case,

$$\langle J\sigma_i, \delta_{x_0} \rangle_{H^s \times H^{-s}} = \sum_{n=1}^{\infty} \lambda_n^{-s} \langle \sigma_i, \omega_n \rangle_{H^s \times H^{-s}} \omega_n(x_0)$$

and

$$\begin{aligned}
\langle J\sigma_i, \sigma_j \rangle_{H^s \times H^{-s}} &= \sum_{n=1}^{\infty} \lambda_n^{-s} \langle \sigma_i, \omega_n \rangle_{H^s \times H^{-s}} \langle \sigma_j, \omega_n \rangle_{H^s \times H^{-s}} \\
&= \langle \sigma_i, \sigma_j \rangle_{H^{-s}}.
\end{aligned}$$

Finally, to approximate $\kappa(x_0)$, we generate the N -vector $\vec{d}(x_0) = [d_i(x_0)] = [\langle J\sigma_i, \delta_{x_0} \rangle_{H^s \times H^{-s}}]$ and the N by N symmetric matrix $[M_{ij}] = \langle \sigma_i, \sigma_j \rangle_{H^{-s}}$, and solve

$$[M_{ij}] [\Phi_j(x_0)] = [d_i(x_0)]. \quad (3.4)$$

3.2 Applying the Backus-Gilbert Method to the Integral Equation. Let us suppose that for admissible input $f(t)$ and an unknown

admissible coefficient $K_1(x)$, the corresponding output $g_1(t)$ in (2.2) has been obtained from experimental measurements. Suppose further that for some arbitrarily chosen admissible coefficient $K^0(x)$, the direct problem (2.1) has been solved and $g_0(t)$ has been computed. Then if we can approximate the function $\kappa(x) = K_1(x) - K^0(x)$, we obtain the approximation $K_1(x) = \kappa(x) + K^0(x)$ for the unknown coefficient. Let us suppose, for convenience, that $K_1(0) = K^0(0)$ and $K_1(1) = K^0(1)$, which is to say $\kappa(0) = \kappa(1) = 0$. Then it is reasonable to consider the eigenfunctions $\{\omega_n(x)\} = \{\sqrt{2} \sin n\pi x\}$ and the eigenvalues $\{\lambda_n^2\} = \{(n\pi)^2\}$ as the orthonormal basis for $H^s(0, 1)$.

Now, the integral identity (2.4 i),

$$\int_0^T \int_0^1 \kappa(x) \partial_x u(x, t, K^0) \partial_x \phi(x, t) dt dx = \int_0^T [g_1(t) - g_0(t)] \phi(1, t) dt,$$

can be expressed as

$$\int_0^1 \kappa(x) \sigma_j(x) dx = \mu_j, \quad j = 1, \dots, N, \quad (3.5)$$

where

$$\begin{aligned} \sigma_j(x) &= \int_0^T \partial_x u(x, t, K^0) \partial_x \phi_j(x, t) dt \\ \mu_j &= \int_0^T [g_1(t) - g_0(t)] \phi_j(1, t) dt. \end{aligned}$$

Here, we are assuming that the adjoint problem (2.3) has been solved for N linearly independent boundary input functions, $\theta_j(t) = \phi_j(1, t)$, and we denote the corresponding N solutions of (2.3) by $\phi_j(x, t)$. Thus, the integral identity provides us with the N moments for the unknown function $\kappa(x)$, and the Backus-Gilbert method requires us to solve (3.4) for the functions $\Phi_j(x_0)$. Then the approximate recovered coefficient is given by

$$K_1(x_0) = K^0(x_0) + \sum_{j=1}^N \Phi_j(x_0) \mu_j.$$

Of course, in order to solve the adjoint problem (2.3) to generate the ingredients of (3.5), it is necessary that the coefficient $K_1(x)$ is known. Consequently, we are forced to introduce an iteration procedure which will converge to $K_1(x)$. For a fixed N and for $n = 0, 1, \dots$

- (i) with coefficient, $K^n(x)$,
- (ii) and data $\theta = \theta_j$, solve (2.3) for ϕ_j for $j = 1, \dots, N$,
- (iii) using ϕ_j compute $\sigma_{j,n}$ and $\mu_{j,n}$,
- (iv) generate M_{ij} and $d_j(x_0)$ and solve (3.4) for $\Phi_{j,n}(x_0)$,

$$(v) K^{n+1}(x_0) = K^n(x_0) + \sum_{j=1}^N \Phi_{j,n}(x_0) \mu_{j,n}.$$

Note that the injectivity of the coefficient to data mapping implies that if $\mu_{j,n} = (g_1(t) - g_n(t), \phi_j(1,t))_{L^2(0,T)}$ tends to zero, then K^n must tend to the true coefficient $K_1(x)$. Note also that once M_{ij} has been generated in step (iv), one can solve for $\Phi_{j,n}(x_0)$ at as many points x_0 in $(0,1)$ as desired, simply by re-evaluating the vector $d_j(x_0)$ at each x_0 . Thus, it is possible to obtain arbitrarily fine spatial resolution with little additional work. The actual resolution parameter is the dimension N of the approximation subspace spanned by the independent vectors $\{\sigma_1, \dots, \sigma_N\}$. The most striking feature of this approach is that a reasonable approximation to the coefficient $K(x)$ will be shown to be obtained with just $N = 1$. This means that the matrix M_{ij} is, in fact, a scalar and (3.4) is solved not by inversion but by simply dividing.

4. Numerical Experiments. Consider the initial boundary value problem (2.1) with $f(t) = t$, and the adjoint problem (2.3) with $\theta(t) = 1-t$. Then the conditions of the lemma are satisfied.

Experiment 1. Let the unknown coefficient be given by

$$K(x) = 2x^2 - x + 1.$$

Then Figure 1 shows the result of choosing an initial guess $K^0(x) = 1+x$ and executing 5 steps of the iteration with $N = 1$ and $s = 3$. Note that the sequence of coefficients $K^n(x)$ converge to the limit $K(x)$ monotonically in this case. The difference sequence printed at the top of the figure are the successive values of $\|g_1(t) - g_n(t)\|_{L^2(0,T)}^2$.

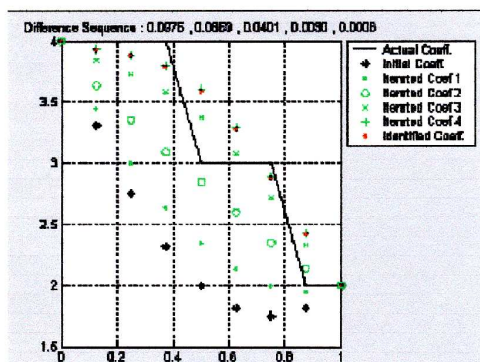


Figure 1. $K(x) = 2x^2 - x + 1$

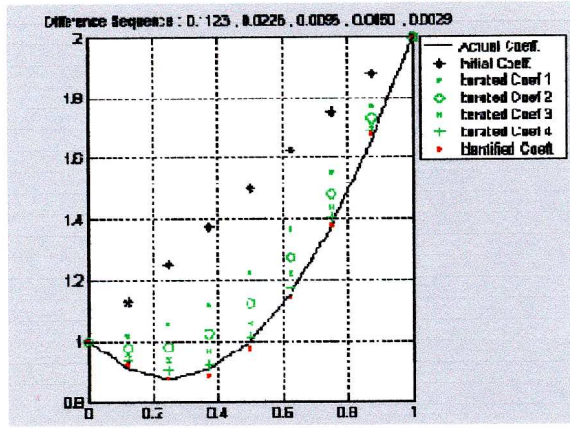


Figure 2. $K(x) = \text{PW linear}$ $N = 1$

Experiment 2. Let the unknown coefficient be the following piecewise linear function,

$$K(x) \begin{cases} 4 & \text{if } 0 \leq x \leq 3/8 \\ 7 - 8x & \text{if } 3/8 \leq x \leq 1/2 \\ 3 & \text{if } 1/2 \leq x < 3/4 \\ 9 - 8x & \text{if } 3/4 \leq x \leq 7/8 \\ 2 & \text{if } 7/8 \leq x < 1. \end{cases}$$

With an initial guess of $K^0(x) = 4x^2 - 6x + 4$, five steps of the iteration process are executed. At this point, the coefficient $K^5(x)$ is a reasonable approximation to $K(x)$ and the approximation is not improved significantly by additional iterations.

In order to obtain better resolution of the unknown coefficient, it would seem to be necessary to increase the value of N . However, numerical experiments have shown no real improvement in the identified coefficient as the result of increasing N to 2 and higher values. Evidently, it will be necessary to find a strategy for choosing the adjoint inputs θ_j so as to produce functions σ_j which span an approximation subspace that better captures the behavior of $K(x)$.

5. Conclusion. This approach to coefficient identification has extremely modest requirements as to the data that must be measured, and the computational complexity of the algorithm to recover the unknown coefficient from this data is similarly modest. In spite of the simplicity, the approach produces a reasonably good approximation to the unknown coefficient in some simple numerical experiments. It remains to be seen how the method will perform when the method is applied in a two dimensional

example and whether a strategy to choose better inputs to the adjoint problem can be devised.

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