A TRANSFORMATION LAW OF AN ETA PRODUCT AND THE INVARIANCE OF A CLASS OF ENTIRE MODULAR FUNCTIONS UNDER $\Gamma_0(\mathbf{n})$

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Abstract. We present a new proof, using Residue Calculus, of the transformation law of a general eta product under $\Gamma_0(n)$ where n is any integer, then we deduce the invariance of a special case of this product under this group and we prove the transformation law of another special case. Our proof is inspired by Siegel's proof [7] of the transformation law of the Dedekind eta function and by Rademacher's generalization [5].

1. Introduction. Let τ be in the upper half plane and $n \in \mathbb{Z}$. The Dedekind eta function is defined by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

Siegel [7] proved the transformation law of the Dedekind eta function under inversion using residue calculus. Rademacher [5] generalized Siegel's method by determining the transformation law of the Dedekind eta function under any element in the full modular group. In this paper, we generalize Rademacher's proof by determining the transformation law of a more general product of eta functions. To define our product, consider

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z,\ c \equiv 0 \pmod{n},\ ad - bc = 1 \right\},$$

a subgroup of the full modular group.
Suppose $n > 1$, and let $\{r_\delta\}$ and $\{r'_\delta\}$ be two sequences of integers indexed by the positive divisors δ of n and suppose that n has g divisors.
Consider the function

$$g_1 = g_1(\tau) = \prod_{l=1}^{g} \frac{\eta(\delta_l \tau)^{r_{\delta_l}}}{\eta(\tau)^{r'_{\delta_l}}}.$$

We prove the transformation law of this function, which is given by

$$g_1(V\tau) = e^{-\pi i \delta^*} \{-i(c\tau + d)\}^{\frac{1}{2}\sum_{l=1}^{g} r_{\delta_l} - \frac{1}{2}\sum_{l=1}^{g} r'_{\delta_l}} g_1(\tau),$$

where

$$\delta^* = \sum_{l=1}^{g} \left\{ \frac{a+d}{12c} + s(-d,c) \right\} r'_{\delta_l} - \sum_{l=1}^{g} \left\{ \frac{a+d}{12c_l} + s(-d,c_l) \right\} r_{\delta_l},$$

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left[\frac{hr}{k} \right] - \frac{1}{2} \right),$$

$$c = c_l \delta_l \text{ and } V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n).$$

A special case of the above product is given by

$$f(\tau) = \prod_{l=1}^{g} \left(\frac{\eta(\delta_l \tau)}{\eta(\tau)} \right)^{r_{\delta_l}}$$

by setting $r_\delta = r'_\delta$ for every δ dividing n in $g(\tau)$.
Imposing certain conditions on r_δ's and r'_δ's will make $f(\tau)$ a function on $\Gamma_0(n)$. Another interesting special case of this product is given by

$$f_1(\tau) = \prod_{l=1}^{g} \eta(\delta_l \tau)^{r_{\delta_l}}$$

by setting $\sum_{\delta | n} r'_\delta = 0$ in $g(\tau)$.
By imposing different conditions this time on r_δ's and r'_δ's, we will deduce a transformation law of $f_1(\tau)$.

1.1 The Transformation Law of $\mathbf{g_1}(\tau)$ Under $\Gamma_\mathbf{0}(\mathbf{n})$. We give a new, detailed proof using residue calculus of the transformation law under $\Gamma_0(n)$.

$$g_1(V\tau) = e^{-\pi i \delta^*} \{-i(c\tau+d)\}^{\frac{1}{2}\sum_{l=1}^{g} r_{\delta_l} - \frac{1}{2}\sum_{l=1}^{g} r'_{\delta_l}} g_1(\tau), \tag{1}$$

where

$$g_1(\tau) = \prod_{l=1}^{g} \frac{\eta(\delta_l \tau)^{r_{\delta_l}}}{\eta(\tau)^{r'_{\delta_l}}}.$$

Consider

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n).$$

Let $a = h'$, $c = k$ and $d = -h$, hence, $k = k_l \delta_l$, where $(h,k) = 1$, $k > 0$, $l = 1, 2, \ldots, g$ and $hh' \equiv -1 \pmod{k}$. We will write $\tau = (h + iz)/k$ and as a result $V\tau = (h' + iz^{-1})/k$.

We have to prove

$$\begin{aligned}
&-\sum_{l=1}^{g}\log\eta\left(\frac{\delta_l h'+i\delta_l z^{-1}}{k}\right)r_{\delta_l}+\sum_{l=1}^{g}\log\eta\left(\frac{\delta_l h+i\delta_l z}{k}\right)r_{\delta_l}\\
&+\sum_{l=1}^{g}\frac{\pi i}{12k_l}(h'-h)r_{\delta_l}+\sum_{l=1}^{g}\pi i s(h,k_l)r_{\delta_l}\\
&+\log\eta\left(\frac{h'+iz^{-1}}{k}\right)\sum_{l=1}^{g}r'_{\delta_l}-\log\eta\left(\frac{h+iz}{k}\right)\sum_{l=1}^{g}r'_{\delta_l}\\
&-\frac{\pi i}{12k}(h'-h)\sum_{l=1}^{g}r'_{\delta_l}-\pi i s(h,k)\sum_{l=1}^{g}r'_{\delta_l}\\
&=-\frac{1}{2}\left(\sum_{l=1}^{g}r_{\delta_l}-\sum_{l=1}^{g}r'_{\delta_l}\right)\log z. \qquad (2)
\end{aligned}$$

The logarithm here is everywhere taken with its principal branch.

Now, from the definition of $\eta(\tau)$,

$$\begin{aligned}
&\log\eta\left(\frac{h+iz}{k}\right)=\frac{\pi i(h+iz)}{12k}+\sum_{m=1}^{\infty}\log(1-e^{2\pi i h m/k}e^{-2\pi z m/k})\\
&=\frac{\pi i(h+iz)}{12k}+\sum_{\mu=1}^{k}\sum_{q=0}^{\infty}\log(1-e^{2\pi i h\mu/k}e^{-2\pi z(qk+\mu)/k})\\
&=\frac{\pi i h}{12k}-\frac{\pi z}{12k}-\sum_{\mu=1}^{k}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i h\mu r/k}e^{-2\pi z(qk+\mu)r/k}\\
&=\frac{\pi i h}{12k}-\frac{\pi z}{12k}-\sum_{\mu=1}^{k}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i h\mu r/k}\frac{e^{-2\pi z\mu r/k}}{1-e^{-2\pi z r}}.
\end{aligned}$$

We substitute the expansion above in (2) and we obtain

$$\begin{aligned}
&\sum_{l=1}^{g}\sum_{\nu=1}^{k_l}\sum_{r=1}^{\infty}\frac{r_{\delta_l}}{r}e^{2\pi ih'\nu r/k_l}\frac{e^{-2\pi\nu r/k_l z}}{1-e^{-2\pi r/z}}\\
&-\sum_{l=1}^{g}\sum_{\mu=1}^{k_l}\sum_{r=1}^{\infty}\frac{r_{\delta_l}}{r}e^{2\pi ih\mu r/k_l}\frac{e^{-2\pi z\mu r/k_l}}{1-e^{-2\pi zr}}\\
&+\sum_{l=1}^{g}\frac{\pi r_{\delta_l}}{12k_l}\left(\frac{1}{z}-z\right)+\pi i\sum_{l=1}^{g}r_{\delta_l}s(h,k_l)\\
&-\sum_{l=1}^{g}\sum_{\nu=1}^{k}\sum_{r=1}^{\infty}\frac{r'_{\delta_l}}{r}e^{2\pi ih'\nu r/k}\frac{e^{-2\pi\nu r/kz}}{1-e^{-2\pi r/z}}\\
&+\sum_{l=1}^{g}\sum_{\mu=1}^{k}\sum_{r=1}^{\infty}\frac{r'_{\delta_l}}{r}e^{2\pi ih\mu r/k}\frac{e^{-2\pi z\mu r/k}}{1-e^{-2\pi zr}}\\
&-\sum_{l=1}^{g}\frac{\pi r'_{\delta_l}}{12k}\left(\frac{1}{z}-z\right)-\pi is(h,k)\sum_{l=1}^{g}r'_{\delta_l}\\
&=-\frac{1}{2}\left(\sum_{l=1}^{g}r_{\delta_l}-\sum_{l=1}^{g}r'_{\delta_l}\right)\log z. \qquad (3)
\end{aligned}$$

To prove (3), we will define a function and calculate the residues of the function at the poles and prove that the sum of the residues is equal to the left side of the above equation. A sort of symmetry is needed between μ and $h\mu$. Therefore, we introduce

$$\mu^* \equiv h\mu \pmod{k}, \qquad (4)$$

where $1 \le \mu^* \le k-1$.
Consider the function

$$\begin{aligned}
F_n(x) = &-\frac{1}{4ix}\coth\pi Nx\cot\frac{\pi Nx}{z}\sum_{l=1}^{g}(r_{\delta_l}-r'_{\delta_l})\\
&+\sum_{l=1}^{g}\sum_{\mu=1}^{k_l-1}\frac{r_{\delta_l}}{x}\cdot\frac{e^{2\pi\mu Nx/k_l}}{1-e^{2\pi Nx}}\cdot\frac{e^{-2\pi i\mu^* Nx/k_l z}}{1-e^{-2\pi iNx/z}}\\
&-\sum_{l=1}^{g}\sum_{\mu=1}^{k_l-1}\frac{r'_{\delta_l}}{x}\cdot\frac{e^{2\pi\mu Nx/k}}{1-e^{2\pi Nx}}\cdot\frac{e^{-2\pi i\mu^* Nx/kz}}{1-e^{-2\pi iNx/z}}, \qquad (5)
\end{aligned}$$

where $N = n+\frac{1}{2}$. We will integrate $F_n(x)$ along the parallelogram with the vertices z, i, $-z$, $-i$ and then calculate the residues of this function at its poles and then compare the two answers using the Residue Theorem.

The function $F_n(x)$ has poles at $x = 0$, $x = ir/N$ and $x = -zr/N$ for $r = \pm 1, \pm 2, \pm 3, \ldots, \pm n$.
The residue at $x = 0$ of the first summand in $F_n(x)$

$$-\frac{1}{4ix} \coth \pi N x \cot \frac{\pi N x}{z} \sum_{l=1}^{g} (r_{\delta_l} - r'_{\delta_l})$$

is

$$-\frac{\sum_{l=1}^{g} (r_{\delta_l} - r'_{\delta_l})}{12i} \left(z - \frac{1}{z} \right). \tag{6}$$

The residue at $x = 0$ of

$$\sum_{l=1}^{g} \frac{r_{\delta_l}}{x} \cdot \frac{e^{2\pi\mu N x/k_l}}{1 - e^{2\pi N x}} \cdot \frac{e^{-2\pi i \mu^* N x/k_l z}}{1 - e^{-2\pi i N x/z}}$$

is

$$\begin{aligned} &\sum_{l=1}^{g} \left(\frac{1}{12} - \frac{\mu}{2k_l} + \frac{1}{2} \frac{\mu^2}{k_l^2} \right) r_{\delta_l} z i + \sum_{l=1}^{g} \left(\frac{\mu}{k_l} - \frac{1}{2} \right) \left(\frac{\mu^*}{k_l} - \frac{1}{2} \right) r_{\delta_l} \\ &+ \sum_{l=1}^{g} \left(\frac{1}{12} - \frac{\mu^*}{2k_l} + \frac{1}{2} \frac{{\mu^*}^2}{k_l^2} \right) \frac{r_{\delta_l}}{iz}. \end{aligned} \tag{7}$$

The second summand of $F_n(x)$ has to be summed over μ from 1 to $k_l - 1$. Observe also that μ^* runs from 1 to $k_l - 1$ for all $l = 1, 2, \ldots, g$ in view of (4). Also, the first and the third summation of (7) are not difficult to calculate. For the middle term, observe from (4) that

$$\frac{\mu^*}{k_l} = \frac{h\mu}{k_l} - \left[\frac{h\mu}{k_l} \right],$$

for $l = 1, 2, 3, \ldots, g$,
so that

$$\sum_{\mu=1}^{k_l - 1} \left(\frac{\mu}{k_l} - \frac{1}{2} \right) \left(\frac{\mu^*}{k_l} - \frac{1}{2} \right) = s(h, k_l).$$

As a result, the residue of the second summand of $F_n(x)$ at $x = 0$ is

$$\begin{aligned} &\sum_{l=1}^{g} \left(\frac{1}{12} - \frac{\mu}{2k_l} + \frac{1}{2} \frac{\mu^2}{k_l^2} \right) r_{\delta_l} z i + \sum_{l=1}^{g} s(h, k_l) r_{\delta_l} \\ &+ \sum_{l=1}^{g} \left(\frac{1}{12} - \frac{\mu^*}{2k_l} + \frac{1}{2} \frac{{\mu^*}^2}{k_l^2} \right) \frac{r_{\delta_l}}{iz}. \end{aligned} \tag{8}$$

The residue of the third summand of the function $F_n(x)$

$$\sum_{l=1}^{g}\sum_{\mu=1}^{k_l-1}\frac{r'_{\delta_l}}{x}\cdot\frac{e^{2\pi\mu Nx/k}}{1-e^{2\pi Nx}}\cdot\frac{e^{-2\pi i\mu^* Nx/kz}}{1-e^{-2\pi iNx/z}}$$

is

$$\sum_{l=1}^{g}\left(\frac{1}{12}-\frac{\mu}{2k}+\frac{1}{2}\frac{\mu^2}{k^2}\right)r'_{\delta_l}zi+\sum_{l=1}^{g}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{\mu^*}{k}-\frac{1}{2}\right)r'_{\delta_l}$$
$$+\sum_{l=1}^{g}\left(\frac{1}{12}-\frac{\mu^*}{2k}+\frac{1}{2}\frac{\mu^{*2}}{k^2}\right)\frac{r'_{\delta_l}}{iz}. \tag{9}$$

Thus, using (6), (8) and (9), we get that the residue at $x=0$ of $F_n(x)$ is

$$\sum_{l=1}^{g}s(h,k_l)r_{\delta_l}+\sum_{l=1}^{g}\frac{ir_{\delta_l}}{12k_l}\left(z-\frac{1}{z}\right)-s(h,k)\sum_{l=1}^{g}r'_{\delta_l}-\sum_{l=1}^{g}\frac{ir'_{\delta_l}}{12k}\left(z-\frac{1}{z}\right).$$

The residue of $F_n(x)$ at $x=\frac{ir}{N}$ is

$$\frac{\sum_{l=1}^{g}(r_{\delta_l}-r'_{\delta_l})}{4\pi r}\cot\frac{\pi ir}{z}-\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu=1}^{k_l-1}\frac{r_{\delta_l}}{r}e^{2\pi i\mu r/k_l}\frac{e^{2\pi\mu^* r/k_l z}}{1-e^{2\pi r/z}}$$
$$+\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu=1}^{k-1}\frac{r'_{\delta_l}}{r}e^{2\pi i\mu r/k}\frac{e^{2\pi\mu^* r/kz}}{1-e^{2\pi r/z}}. \tag{10}$$

It is easy to see that

$$h'\mu\equiv hh'\mu\equiv-\mu\pmod{k_l},$$

for $l=1,2,3,\ldots,g$, and

$$h'\mu\equiv hh'\mu\equiv-\mu\pmod{k}.$$

As a result, (10) becomes

$$\frac{\sum_{l=1}^{g}(r_{\delta_l}-r'_{\delta_l})}{4\pi ir}\coth\frac{\pi r}{z}-\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu=1}^{k_l-1}\frac{r_{\delta_l}}{r}e^{-2\pi ih'\mu^* r/k_l}\frac{e^{2\pi\mu^* r/k_l z}}{1-e^{2\pi r/z}}$$
$$+\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu=1}^{k-1}\frac{r'_{\delta_l}}{r}e^{-2\pi ih'\mu^* r/k}\frac{e^{2\pi\mu^* r/kz}}{1-e^{2\pi r/z}}.$$

The parallelogram contains the poles $x = \frac{ir}{N}$ for $-n \le r \le -1$ and $1 \le r \le n$. Then, we sum over the poles and obtain

$$\frac{\sum_{l=1}^{g}(r_{\delta_l} - r_{\delta_l'})}{2\pi i}\sum_{r=1}^{n}\frac{1}{r}\left(\frac{2e^{-2\pi r/z}}{1-e^{-2\pi r/z}}+1\right)$$
$$+\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu^*=1}^{k_l-1}\sum_{r=1}^{n}\frac{r_{\delta_l}}{r}e^{2\pi i h'\mu^* r/k_l}\frac{e^{-2\pi\mu^* r/k_l z}}{1-e^{-2\pi r/z}}$$
$$-\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu^*=1}^{k_l-1}\sum_{r=1}^{n}\frac{r_{\delta_l}}{r}e^{2\pi i h'(k_l-\mu^*) r/k_l}\frac{e^{-2\pi(k_l-\mu^*) r/k_l z}}{e^{-2\pi r/z}-1}$$
$$-\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu^*=1}^{k-1}\sum_{r=1}^{n}\frac{r'_{\delta_l}}{r}e^{2\pi i h'\mu^* r/k}\frac{e^{-2\pi\mu^* r/kz}}{1-e^{-2\pi r/z}}$$
$$+\frac{1}{2\pi i}\sum_{l=1}^{g}\sum_{\mu^*=1}^{k-1}\sum_{r=1}^{n}\frac{r'_{\delta_l}}{r}e^{2\pi i h'(k-\mu^*) r/k}\frac{e^{-2\pi(k-\mu^*) r/kz}}{e^{-2\pi r/z}-1}.$$

In the third and fifth summands of the above sum, we replace $k_l - \mu^*$ and $k - \mu^*$ by μ^* and combine it with the other sum. As a result, the residue of $F_n(x)$ at $x = \frac{ir}{N}$ is given by

$$\frac{\sum_{l=1}^{g}(r_{\delta_l} - r_{\delta_l'})}{2\pi i}\sum_{r=1}^{n}\frac{1}{r}+\frac{1}{\pi i}\sum_{l=1}^{g}\sum_{\nu=1}^{k_1}\sum_{r=1}^{n}\frac{r_{\delta_l}}{r}e^{2\pi i h'\nu r/k_l}\frac{e^{-2\pi\nu r/k_l z}}{1-e^{-2\pi r/z}}$$
$$-\frac{1}{\pi i}\sum_{l=1}^{g}\sum_{\nu=1}^{k}\sum_{r=1}^{n}\frac{r'_{\delta_l}}{r}e^{2\pi i h'\nu r/k}\frac{e^{-2\pi\nu r/kz}}{1-e^{-2\pi r/z}}.$$

Similarly, we find the sum of the residues of $F_n(x)$ at $x = -\frac{zr}{N}$, $r = \pm1, \pm2, \pm3, \dots, \pm n$ is

$$\frac{i\sum_{l=1}^{g}(r_{\delta_l} - r'_{\delta_l})}{2\pi}\sum_{r=1}^{n}\frac{1}{r}+\frac{i}{\pi}\sum_{l=1}^{g}\sum_{\nu=1}^{k_l}\sum_{r=1}^{n}\frac{r_{\delta_l}}{r}e^{2\pi i h\nu r/k_l}\frac{e^{-2\pi\nu r z/k_l}}{1-e^{-2\pi r z}}$$
$$-\frac{i}{\pi}\sum_{l=1}^{g}\sum_{\nu=1}^{k}\sum_{r=1}^{n}\frac{r'_{\delta_l}}{r}e^{2\pi i h\nu r/k}\frac{e^{-2\pi\nu r z/k}}{1-e^{-2\pi r z}}.$$

Thus, the sum of all the residues of $F_n(x)$ within the parallelogram is

$$\begin{aligned}
&\sum_{l=1}^{g} \frac{r_{\delta_l}}{12k_l i}\left(\frac{1}{z}-z\right)+\sum_{l=1}^{g} s(h,k_l) r_{\delta_l} \\
&+\frac{1}{\pi i}\sum_{l=1}^{g}\sum_{\nu=1}^{k_l}\sum_{r=1}^{n} \frac{r_{\delta_l}}{r} e^{2\pi i h'\nu r/k_l} \frac{e^{-2\pi\nu r/k_l z}}{1-e^{-2\pi r/z}} \\
&-\frac{1}{\pi i}\sum_{l=1}^{g}\sum_{\mu=1}^{k_l}\sum_{r=1}^{n} \frac{r_{\delta_l}}{r} e^{2\pi i h\mu r/k_l} \frac{e^{-2\pi\mu r z/k_l}}{1-e^{-2\pi r z}} \\
&-\sum_{l=1}^{g} \frac{r'_{\delta_l}}{12ki}\left(\frac{1}{z}-z\right)-\sum_{l=1}^{g} s(h,k) r'_{\delta_l} \\
&-\frac{1}{\pi i}\sum_{l=1}^{g}\sum_{\nu=1}^{k}\sum_{r=1}^{n} \frac{r'_{\delta_l}}{r} e^{2\pi i h'\nu r/k} \frac{e^{-2\pi\nu r/kz}}{1-e^{-2\pi r/z}} \\
&+\frac{1}{\pi i}\sum_{l=1}^{g}\sum_{\mu=1}^{k}\sum_{r=1}^{n} \frac{r'_{\delta_l}}{r} e^{2\pi i h\mu r/k} \frac{e^{-2\pi\mu r z/k}}{1-e^{-2\pi r z}}.
\end{aligned}$$

What remains to prove is that

$$\lim_{n\to\infty} \int_C F_n(x)dx = -\left(\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l}\right)\log z,$$

where C is the parallelogram of vertices z, i, $-z$, $-i$.
Now on the four sides of C, except at the vertices, the second and the third summands in $F_n(x)$ goes to zero as n goes to infinity. Regarding the first part of the function, it is easy to see that

$$\lim_{n\to\infty} \coth \pi N x \cot \frac{\pi N x}{z} = i$$

on the sides i to $-z$ and $-i$ to z and that

$$\lim_{n\to\infty} \coth \pi N x \cot \frac{\pi N x}{z} = -i$$

on the sides i to z and $-i$ to $-z$. Therefore,

$$\lim_{n\to\infty} F_n(x) = \frac{\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l}}{4}$$

on the sides i to z and on $-i$ to $-z$, and

$$\lim_{n\to\infty} F_n(x) = -\frac{\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l}}{4}$$

on the sides i to $-z$ and on $-i$ to z. The convergence of $F_n(x)$ is not uniform but it is bounded, since the denominators of the three summands are bounded away from zero and this is because $N = n + \frac{1}{2}$ is not an integer. We then have

$$
\begin{aligned}
&\lim_{n\to\infty} \int_C F_n(x)dx \\
&= \frac{\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l}}{4} \left\{ -\int_{-i}^{z} \frac{dx}{x} + \int_{z}^{i} \frac{dx}{x} - \int_{i}^{-z} \frac{dx}{x} + \int_{-z}^{-i} \frac{dx}{x} \right\} \\
&= \frac{\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l}}{2} \left\{ -\int_{-i}^{z} \frac{dx}{x} + \int_{z}^{i} \frac{dx}{x} \right\} \\
&= \frac{\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l}}{2} \left\{ -\left(\log z + \frac{\pi i}{2} \right) + \left(\frac{\pi i}{2} - \log z \right) \right\} \\
&= -\left(\sum_{l=1}^{g} r_{\delta_l} - r'_{\delta_l} \right) \log z.
\end{aligned}
$$

1.2 A Special Case of $\mathbf{g_1}(\tau)$. Let

$$f(\tau) = \prod_{l=1}^{g} \left(\frac{\eta(\delta_l \tau)}{\eta(\tau)} \right)^{r_{\delta_l}}.$$

Also, suppose that

$$\frac{1}{24} \sum_{l=1}^{g} (\delta_l - 1) r_{\delta_l} \tag{11}$$

is an integer and

$$\frac{1}{24} \sum_{l=1}^{g} (\delta'_l - n) r_{\delta_l} \tag{12}$$

is an integer, where $n = \delta_l \delta'_l$. Also, assume that

$$\prod_{l=1}^{g} \delta_l^{r_{\delta_l}} \tag{13}$$

is a rational square and that $r_1 = 0$.

It is easy to see that $f(\tau)$ is the special case of $g_1(\tau)$, in which $r_\delta = r'_\delta$ for all δ dividing n. We then have

$$f(V\tau) = e^{-\pi i \delta^{**}} f(\tau),$$

where

$$\delta^{**} = \sum_{l=1}^{g} \left\{ \left\{ \frac{a+d}{12c} + s(-d,c) \right\} - \left\{ \frac{a+d}{12c_l} + s(-d,c_l) \right\} \right\} r_{\delta_l}.$$

Suppose now that $(a,6)=1$ and $c>0$. M. Newman [2] using (11), (12) and (13) showed that

$$\sum_{l=1}^{g} \left\{ \left\{ \frac{a+d}{12c} + s(-d,c) \right\} - \left\{ \frac{a+d}{12c_l} + s(-d,c_l) \right\} \right\} r_{\delta_l}$$

is an even integer. Hence,

$$f(V\tau) = f(\tau),$$

where $V \in \Gamma_0(n)$.

In [2], M. Newman mentioned that since $S = \tau+1$ is in $\Gamma_0(n)$ for every n, $\Gamma_0(n)$ can be generated by the elements

$$\begin{pmatrix} a & b \\ nc & d \end{pmatrix},$$

where $(a,6)=1$. Thus, it is necessary to show the invariance of a function only with respect to these transformations in order to show its invariance for $\Gamma_0(n)$. Also, it suffices to consider only these substitutions for which both a and nc are positive.

1.3 Another Special Case of $\mathbf{g_1}(\tau)$.

Let

$$f_1(\tau) = \prod_{l=1}^{g} \eta(\delta_l \tau)^{r_{\delta_l}},$$

where

$$\sum_{l=1}^{g} \delta_l r_{\delta_l} \equiv 0 \pmod{24} \tag{14}$$

and

$$\sum_{l=1}^{g} \frac{n}{\delta_l} r_{\delta_l} \equiv 0 \pmod{24}. \tag{15}$$

Let $k = \frac{1}{2}\sum_{l=1}^{g} r_{\delta_l} \in \mathbb{Z}$. It is easy to see that $f_1(\tau)$ is a special case of $g_1(\tau)$, where $\sum_{l=1}^{g} r'_{\delta_l} = 0$. We then have

$$f_1(V\tau) = e^{-\pi i \delta^{***}} \{-i(c\tau+d)\}^k f_1(\tau),$$

where

$$\delta^{***} = \sum_{l=1}^{g} \left\{ -\frac{a+d}{12c_l} - s(-d, c_l) \right\} r_{\delta_l}.$$

We have to prove now that the transformation law above is the same as

$$f_1(V\tau) = \chi(d)(c\tau + d)^k f_1(\tau),$$

where $V \in \Gamma_0(n)$ and

$$\chi(d) = \left(\frac{(-1)^k \prod_{l=1}^{g} \delta_l^{r_{\delta_l}}}{d} \right).$$

Since k is an integer, we get

$$f_1(V\tau) = e^{-\pi i \delta^{***}} (-i)^k (c\tau + d)^k f_1(\tau).$$

What remains to prove is that

$$\chi(d) = (-i)^k e^{-\pi i \delta^{***}}.$$

Notice that $-ad \equiv -1 \pmod{c}$. Thus, $s(-d, c) = -s(a, c)$.

We have that

$$\delta M\tau = \delta \begin{pmatrix} a & b \\ nc_1 & d \end{pmatrix} \tau = \begin{pmatrix} a & \delta b \\ \delta' c_1 & d \end{pmatrix} \delta\tau = M_1 \delta\tau,$$

where $M \in \Gamma_0(n)$.
Thus, $\eta(\delta M\tau) = \eta(M_1\delta\tau)$ and so

$$f(M\tau) = \prod_{l=1}^{g} \eta(\delta_l M\tau)^{r_{\delta_l}} = \prod_{l=1}^{g} \eta(M_1 \delta_l \tau)^{r_{\delta_l}}.$$

Assume now that $(a, 6) = 1$, $c > 0$ and $n = \delta_l \delta_l'$. In [3], Newman proved that

$$s(a, c) - (a+d)/12c \equiv \frac{1}{12} a(c - b - 3) - \frac{1}{2} \left\{ 1 - \left(\frac{c}{a} \right) \right\} \pmod{2},$$

where $\left(\frac{c}{a}\right)$ is the generalized Legendre-Jacobi symbol of the quadratic reciprocity. Write $c = c_1 n$. Thus,

$$\begin{aligned}
\delta^{***} &= \sum_{l=1}^{g} \left\{ s(a, \delta_l' c_1) - \frac{(a+d)}{12\delta_l' c_1} \right\} r_{\delta_l} \\
&\equiv \frac{ac_1}{12} \sum_{l=1}^{g} \delta_l' r_{\delta_l} - \frac{ab}{12} \sum_{l=1}^{g} \delta_l r_{\delta_l} \\
&\quad - \frac{3a}{12} \sum_{l=1}^{g} r_{\delta_l} - \frac{1}{2} \sum_{l=1}^{g} \left\{ 1 - \left(\frac{\delta_l' c_1}{a} \right) \right\} r_{\delta_l} \pmod{2} \\
&\equiv -\frac{2ac_1}{24} \sum_{l=1}^{g} \delta_l' r_{\delta_l} + \frac{2ab}{24} \sum_{l=1}^{g} \delta_l r_{\delta_l} + \frac{k}{2} + \frac{1}{2} \sum_{l=1}^{g} \left\{ 1 - \left(\frac{\delta_l}{a} \right) \right\} r_{\delta_l} \pmod{2}.
\end{aligned}$$

But we are given that $\sum_{l=1}^{g} \delta_l' r_{\delta_l} \equiv 0 \pmod{24}$ and $\sum_{l=1}^{g} \delta_l r_{\delta_l} \equiv 0 \pmod{24}$. Thus,

$$\frac{ac_1}{12} \sum_{l=1}^{g} \delta_l' r_{\delta_l}$$

and

$$\frac{ab}{12} \sum_{l=1}^{g} \delta_l r_{\delta_l}$$

are even integers. Therefore, we get

$$\begin{aligned} e^{-\pi i \delta^{***}} &= e^{\pi i \frac{1}{2} k} e^{\pi i \frac{1}{2} \sum_{l=1}^{g} \left\{1 - \left(\frac{\delta_l}{a}\right)\right\} r_{\delta_l}} \\ &= (-i)^k e^{\pi i \frac{1}{2} \sum_{l=1}^{g} \left\{1 - \left(\frac{\delta_l}{a}\right)\right\} r_{\delta_l}}. \end{aligned}$$

Now,

$$e^{\pi i \frac{1}{2} \sum_{l=1}^{g} \left\{1 - \left(\frac{\delta_l}{a}\right)\right\} r_{\delta_l}} = \prod_{l=1}^{g} \left(\frac{\delta_l}{a}\right)^{r_{\delta_l}}.$$

Thus,

$$(-i)^k e^{-\pi i \delta^{***}} = \left(\frac{(-1)^k \prod_{l=1}^{g} \delta_l^{r_{\delta_l}}}{a}\right).$$

But $ad - bc = 1$, as a result we get

$$(-i)^k e^{-\pi i \delta^{***}} = \left(\frac{(-1)^k \prod_{l=1}^{g} \delta_l^{r_{\delta_l}}}{d}\right),$$

and hence,

$$f_1(V\tau) = \chi(d)(c\tau + d)^k f_1(\tau),$$

where

$$\chi(d) = \left(\frac{(-1)^k \prod_{l=1}^{g} \delta_l^{r_{\delta_l}}}{d}\right).$$

Acknowledgement. We would like to thank our Ph.D. advisor Professor Marvin Knopp for his guidance. We would also like to thank the referee for the helpful comments.

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Mathematics Subject Classification (2000): 11F11, 11F20

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