No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

165. [2007; 151] *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\frac{1}{2n}\left(\sum_{k=1}^{n}\sqrt{1+\left(F_k\binom{n}{k}\right)^2}\right)^2 \geq F_{2n},$$

where F_n is the nth Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Solution I by Brian Bradie, Christopher Newport University, Newport News, Virginia. For $n = 1$, we have

$$\frac{1}{2}\left(\sqrt{1+F_1^2}\right)^2 = \frac{1}{2}\left(\sqrt{1+1}\right)^2 = 1 = F_2;$$

whereas, for $n = 2$, we have

$$\begin{aligned}\frac{1}{4}\left(\sqrt{1+(2F_1)^2}+\sqrt{1+F_2^2}\right)^2 &= \frac{1}{4}\left(\sqrt{1+4}+\sqrt{1+1}\right)^2\\ &= \frac{1}{4}\left(7+2\sqrt{10}\right) > 3 = F_4.\end{aligned}$$

To proceed further, first note that by a straightforward application of mathematical induction we can show that $F_m > m$ for all $m \geq 6$. Now, let $n \geq 3$. Then

$$\frac{1}{2n}\left(\sum_{k=1}^{n}\sqrt{1+\left(F_k\binom{n}{k}\right)^2}\right)^2 > \frac{1}{2n}\left(\sum_{k=1}^{n}F_k\binom{n}{k}\right)^2 = \frac{1}{2n}F_{2n}^2,$$

where we have used the identity

$$\sum_{k=1}^{n} F_k \binom{n}{k} = F_{2n}.$$

Because $n \geq 3$, it follows that $2n \geq 6$, so that $F_{2n} > 2n$. Therefore,

$$\frac{1}{2n}\left(\sum_{k=1}^{n}\sqrt{1+\left(F_k\binom{n}{k}\right)^2}\right)^2 > \frac{F_{2n}}{2n}\cdot F_{2n} > F_{2n},$$

as desired.

Solution II by Joe Howard, Portales, New Mexico. We prove a stronger result.

Claim. For nonnegative real numbers a_i, $1 \leq i \leq n$,

$$\sum_{i=1}^{n}\left(1+a_i^2\right)^{1/2} \geq \left(n^2+\left(\sum_{i=1}^{n}a_i\right)^2\right)^{1/2} \geq \left(2n\sum_{i=1}^{n}a_i\right)^{1/2}.$$

Proof. To prove the first inequality, we note that if $f(x) = (1+x^2)^{1/2}$, then

$$f'(x) = \frac{x}{(1+x^2)^{1/2}} \quad \text{and} \quad f''(x) = \frac{1}{(1+x^2)^{3/2}} > 0,$$

so f is convex. By Jensen's inequality, it follows that

$$\frac{1}{n}\sum_{i=1}^{n}(1+a_i^2)^{1/2} \geq \left(1+\left(\frac{\sum_{i=1}^{n}a_i}{n}\right)^2\right)^{1/2}.$$

Therefore,

$$\sum_{i=1}^{n}(1+a_i^2)^{1/2} \geq \left(n^2+\left(\sum_{i=1}^{n}a_i\right)^2\right)^{1/2}.$$

To prove the second inequality, note that the following three inequalities are equivalent:

$$n^2 + \left(\sum_{i=1}^{n} a_i\right)^2 \geq 2n \sum_{i=1}^{n} a_i$$

$$n^2 - 2n \sum_{i=1}^{n} a_i + \left(\sum_{i=1}^{n} a_i\right)^2 \geq 0$$

$$\left(n - \sum_{i=1}^{n} a_i\right)^2 \geq 0.$$

Since the last inequality is obviously true, the second inequality is true.

Finally, using the result

$$\sum_{k=1}^{n} F_k \binom{n}{k} = F_{2n},$$

which can be found on p. 51 of *Fibonacci and Lucas Numbers* by V. E. Hoggatt, we obtain the stronger result:

$$\left(\sum_{k=1}^{n} \sqrt{1 + \left(F_k \binom{n}{k}\right)^2}\right)^2 \geq n^2 + F_{2n}^2.$$

Also solved by the proposer.

166. [2007; 151] *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan.*

Find the sum:

$$\sum_{n=1}^{\infty} (-1)^n \left(\ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n}\right).$$

Solution by Panagiotis T. Krasopoulos, Athens, Greece. We define

$$S_m = \sum_{n=1}^{m} a_n,$$

where

$$a_n = (-1)^n \left(\ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right).$$

Thus, we want to calculate $\lim_{m\to\infty} S_m$. Next, we have

$$\begin{aligned} a_n + a_{n+1} &= (-1)^n \left(-\frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} \right) \\ &= \frac{1}{2}(-1)^{n+1} \left(\frac{1}{n+1} - \frac{2}{2n+1} \right). \end{aligned}$$

We define

$$R_m = \sum_{n=1}^{m} (a_n + a_{n+1}) = S_m + S_{m+1} - a_1$$

and we have that

$$\begin{aligned} R_m &= \frac{1}{2} \sum_{n=1}^{m} \left((-1)^{n+1} \left(\frac{1}{n+1} - \frac{2}{2n+1} \right) \right) \\ &= \frac{1}{2} \sum_{n=1}^{m} \left((-1)^{n+1} \frac{1}{n+1} \right) - \sum_{n=1}^{m} \left((-1)^{n+1} \frac{1}{2n+1} \right). \end{aligned}$$

Next, we use the following Taylor formulas:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots ,$$

and for $x = 1$ we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots ,$$

and for $x = 1$ we get

$$\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

From these formulas we get

$$\lim_{m\to\infty} R_m = \frac{1}{2}(1 - \ln 2) - \left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2} - \frac{1}{2}\ln 2.$$

Finally, if we consider the limits as $m \to \infty$ we have

$$\begin{aligned}\lim_{m\to\infty} S_m &= \frac{1}{2}\left(a_1 + \lim_{m\to\infty} R_m\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \ln 2 + \frac{\pi}{4} - \frac{1}{2} - \frac{1}{2}\ln 2\right) \\ &= \frac{\pi}{8} - \frac{3}{4}\ln 2.\end{aligned}$$

This completes the proof.

Also solved by Joe Howard, Portales, New Mexico; Brian Bradie, Christopher Newport University, Newport News, Virginia; Huizeng Qin, Shandong University of Technology, Zibo, Shadong, People's Republic of China and Youmin Lu, Bloomsburg University, Bloomsburg, Pennsylvania (jointly); Kenneth B. Davenport, Dallas, Pennsylvania; Paolo Perfetti, Università degli studi "Tor Vergata", Roma, Italy; Kee-Wai Lau, Hong Kong, China; and the proposer.

167. [2007; 152] *Proposed by Victor Dontsov, Evgeni Maevski and Zokhrab Mustafaev, University of Houston-Clear Lake, Houston, Texas.*

Show that there is a continuous function $f(x)$ on $[0, \pi]$ that satisfies the functional equation

$$f(x) + \frac{1}{2}\sin f(x) = x \quad \text{and find} \quad \int_0^\pi f(x)dx.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, Virginia.

Observe that the equation

$$f + \frac{1}{2}\sin f = x \tag{1}$$

explicitly definies x as a function of f. This function is continuous on the real line; moreover,

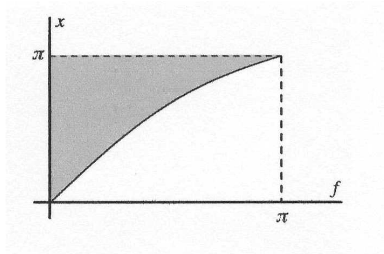
$$\frac{dx}{df} = 1 + \frac{1}{2}\cos f \geq \frac{1}{2}$$

for all f. Therefore, by the Implicit Function Theorem, equation (1) implicitly defines a function $f(x)$ that is continuous and differentiable on the real line, not just $[0, \pi]$. Now, the definite integral

$$\int_0^\pi f(x)dx$$

represents the area of the shaded region shown in the figure below, where the solid curve is the graph of equation (1). From the geometry in the figure, we see that

$$\begin{aligned}\int_0^\pi f(x)dx &= \pi^2 - \int_0^\pi \left(f + \frac{1}{2}\sin f\right) df \\ &= \pi^2 - \left(\frac{1}{2}\pi^2 + 1\right) \\ &= \frac{1}{2}\pi^2 - 1.\end{aligned}$$



Also solved by Huizeng Qin, Shandong University of Technology, Zibo, Shadong, People's Republic of China and Youmin Lu, Bloomsburg University, Bloomsburg, Pennsylvania (2 solutions, jointly), and the proposer. One partial solution was also received.

168. [2007; 152] *Proposed by Don Redmond, Southern Illinois University at Carbondale, Carbondale, Illinois.*

Let $N \geq 2$ be an integer. Evaluate the product

$$\prod_{k=1}^{N^2-1} \Gamma\left(\frac{k}{N}\right).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, Virginia. We first rewrite the product as

$$\prod_{k=1}^{N^2-1} \Gamma\left(\frac{k}{N}\right) = \prod_{k=1}^{N-1} \Gamma\left(\frac{k}{N}\right) \cdot \prod_{\ell=1}^{N-1} \left(\prod_{k=0}^{N-1} \Gamma\left(\ell + \frac{k}{N}\right)\right),$$

and then use the Gauss Multiplication Formula:

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right) \cdot \cdots \cdot \Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{(n-1)/2} n^{\frac{1}{2}-nz} \Gamma(nz).$$

With $z = \frac{1}{N}$ and $n = N$, we find

$$\prod_{k=1}^{N-1} \Gamma\left(\frac{k}{N}\right) = (2\pi)^{(N-1)/2} N^{-1/2},$$

while, with $z = \ell$ and $n = N$, we find

$$\prod_{k=0}^{N-1} \Gamma\left(\ell + \frac{k}{N}\right) = (2\pi)^{(N-1)/2} N^{\frac{1}{2} - N\ell} \Gamma(N\ell).$$

Thus,

$$\begin{aligned}
\prod_{k=1}^{N^2-1} \Gamma\left(\frac{k}{N}\right) &= (2\pi)^{(N-1)/2} N^{-1/2} \prod_{\ell=1}^{N-1} (2\pi)^{(N-1)/2} N^{\frac{1}{2} - N\ell} \Gamma(N\ell) \\
&= (2\pi)^{N(N-1)/2} N^{(-N^3+N^2+N-2)/2} \prod_{\ell=1}^{N-1} \Gamma(N\ell) \\
&= (2\pi)^{N(N-1)/2} N^{(-N^3+N^2+N-2)/2} \prod_{\ell=1}^{N-1} (N\ell - 1)!.
\end{aligned}$$

Also solved by Armend Shabani, University of Prishtina, Prishtina, Kosavo; Panagiotis T. Krasopoulos, Athens, Greece; Joe Howard, Portales, New Mexico; Paolo Perfetti, Universitá degli studi "Tor Vergata", Roma, Italy; and the proposer.