IDEALS IN DORROH EXTENSIONS OF RINGS

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Abstract. The Dorroh extension is typically applied to embed a ring without unity into a ring containing unity. However, de Alwis investigated the ring resulting from applying this extension to the ring of integers. We show that the Dorroh extension of any ring with unity is isomorphic to a direct product of rings. Using this isomorphism we are able to verify the results of de Alwis and extend them to the Dorroh extension of any ring with unity. Also, given any ring R, we give conditions under which an ideal of the Dorroh extension $\mathbb{Z} * R$ can be expressed as a product (in the extension) of an ideal in $\mathbb{Z}$ and an ideal in R.

1. Introduction. Let R be a ring and let $\mathbb{Z}$ denote the set of integers. A common method for embedding R into a ring with identity is via the Dorroh extension. On the underlying set $\mathbb{Z} \times R$, define addition and multiplication by $(z_1, r_1) + (z_2, r_2) = (z_1 + z_2, r_1 + r_2)$ and $(z_1, r_1) * (z_2, r_2) = (z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2)$. Then $(\mathbb{Z} \times R, +, *)$ is a ring with identity $(1, 0)$. We denote this ring by $\mathbb{Z} * R$.

In [1], the ring $\mathbb{Z} * \mathbb{Z}$ is investigated with the invertible elements, zero divisors, and prime and maximal ideals being identified. We extend these results to $\mathbb{Z} * R$, where R is any ring with identity. Throughout the paper, for subrings I and J of $\mathbb{Z}$ and R, respectively, $I * J$ will denote the subring $(I \times J, +, *)$. For any two rings S and T, $S \times T$ will denote the Cartesian product ring with the usual componentwise addition and multiplication.

2. Ideals in $\mathbb{Z} * \mathbf{R}$. It is an exercise to show that S is an ideal of $\mathbb{Z} \times R$ if and only if $S = I \times J$, where I is an ideal of $\mathbb{Z}$ and J is an ideal of R. However, the same is not true in $\mathbb{Z} * R$. De Alwis [1] shows that $\langle (1, 1) \rangle$ is an ideal in $\mathbb{Z} * \mathbb{Z}$ that cannot be written in the form $I * J$, where I and J are ideals of $\mathbb{Z}$. We first determine when $I * J$ is an ideal of $\mathbb{Z} * R$.

Theorem 2.1. Let R be a ring, not necessarily with unity, and let I be an ideal of $\mathbb{Z}$ and J be an ideal of R. Then $I * J$ is an ideal of $\mathbb{Z} * R$ if and only if $ir \in J$ for all $i \in I$ and $r \in R$.

Proof. Assume that $I * J$ is an ideal of $\mathbb{Z} * R$. Let $(i, j) \in I * J$ and $(z, r) \in \mathbb{Z} * R$. Then $(i, j) * (z, r) = (iz, ir + zj + jr) \in I * J$ and $ir + zj + jr \in J$. Since J is an ideal of R, then $zj \in J$ and $jr \in J$. It follows that $ir \in J$. The converse is now clear.

Theorem 2.2. Let R be a ring with identity 1_R and let I be an ideal of $\mathbb{Z}$ and J be an ideal of R. If $I * J$ is an ideal of $\mathbb{Z} * R$, then $(\mathbb{Z} * R)/(I * J) \cong \mathbb{Z}/I \times R/J$.

Proof. Define a mapping $\varphi\colon \mathbb{Z} * R \to \mathbb{Z}/I \times R/J$ by $\varphi(z,r) = (z + I, z \cdot 1_R + r + J)$. Then $\varphi[(z_1, r_1) + (z_2, r_2)] = \varphi(z_1 + z_2, r_1 + r_2) = (z_1 + z_2 + I, (z_1 + z_2) \cdot 1_R + r_1 + r_2 + J) = (z_1 + I, z_1 \cdot 1_R + r_1 + J) + (z_2 + I, z_2 \cdot 1_R + r_2 + J) = \varphi(z_1, r_1) + \varphi(z_2, r_2)$ and φ is a group homomorphism. Also, $\varphi[(z_1, r_1) * (z_2, r_2)] = \varphi(z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2) = (z_1 z_2 + I, (z_1 z_2) \cdot 1_R + z_1 r_2 + z_2 r_1 + r_1 r_2 + J) = (z_1 + I, z_1 \cdot 1_R + r_1 + J) \cdot (z_2 + I, z_2 \cdot 1_R + r_2 + J) = \varphi(z_1, r_1) \cdot \varphi(z_2, r_2)$, and φ is a ring homomorphism. Finally, let $(z + I, r + J) \in \mathbb{Z}/I \times R/J$ and consider $(z, -z \cdot 1_R + r) \in \mathbb{Z} * R$. Then $\varphi(z, -z \cdot 1_R + r) = (z + I, z \cdot 1_R + -z \cdot 1_R + r + J) = (z + I, r + J)$, and φ is a surjection.

If (z, r) is in the kernel of φ, then $\varphi(z,r) = (z+I, z\cdot 1_R + r + J) = (I, J)$. Thus, $z + I = I$ and $z \in I$. Since $I * J$ is an ideal of $\mathbb{Z} * R$, then by the previous theorem, $z \cdot 1_R \in J$. Since $z \cdot 1_R + r + J = J$, it follows that $r \in J$. So the kernel of φ is a subset of $I * J$. Using the above theorem again, for $(i, j) \in I * J$, $\varphi(i, j) = (i + I, i \cdot 1_R + j + J) = (I, J)$. So the kernel of φ is $I * J$. Since φ is a surjection with kernel $I * J$, then by the first isomorphism theorem, $(\mathbb{Z} * R)/(I * J) \cong \mathbb{Z}/I \times R/J$.

Corollary 2.3. Let R be a ring with identity. Then $\mathbb{Z} * R \cong \mathbb{Z} \times R$.

Proof. Consider the ideals $I = \{0\}$ of $\mathbb{Z}$ and $J = \{0\}$ of R. Then $ir \in J$ for all $i \in I$ and $r \in R$ and $I * J$ is an ideal of $\mathbb{Z} * R$. By Theorem 2.2, $\mathbb{Z} * R \cong (\mathbb{Z} * R)/(I * J) \cong \mathbb{Z}/I \times R/J \cong \mathbb{Z} \times R$.

Since the isomorphism above reduces the study of $\mathbb{Z} * R$ to that of $\mathbb{Z} \times R$, we collect some results about the direct product $\mathbb{Z} \times R$. The proofs of these results are typical undergraduate exercises; as such, they are left to the reader.

Theorem 2.4. Let R be a ring.

1. All ideals of $\mathbb{Z} \times R$ are of the form $I \times J$, where I is an ideal of $\mathbb{Z}$ and J is an ideal of R.
2. All prime ideals of $\mathbb{Z} \times R$ are of the form $\mathbb{Z} \times P$, $p\mathbb{Z} \times R$, or $\{0\} \times R$, where P is a prime ideal of R and $p \in \mathbb{Z}$ is prime.
3. All maximal ideals of $\mathbb{Z} \times R$ are of the form $\mathbb{Z} \times M$ or $p\mathbb{Z} \times R$, where M is a maximal ideal of R and $p \in \mathbb{Z}$ is prime.
4. If R has an identity, then the invertible elements of $\mathbb{Z} \times R$ are of the form $(\pm 1, u)$, where u is a unit in R.
5. The zero divisors of $\mathbb{Z} \times R$ are of the form $(z, 0)$ or $(0, r)$ for any $z \in \mathbb{Z}$ and $r \in R$, or (z, s), where $z \in \mathbb{Z}$ and s is a zero divisor in R.

By combining the previous two results, one can determine the ideals, prime ideals, maximal ideals, invertible elements, and zero divisors of $\mathbb{Z} * R$ using the corresponding results in $\mathbb{Z} \times R$. Letting $R = \mathbb{Z}$, we verify the following results given in [1].

Theorem 2.5. Consider the ring $\mathbb{Z} * \mathbb{Z}$.

1. All prime ideals of $\mathbb{Z} * \mathbb{Z}$ are given by $I_1 = \langle(0,1)\rangle$, $I_2 = \langle(1,-1)\rangle$, $I_3 = \langle(1,-1+p)\rangle$, and $I_4 = \langle(p,1-p)\rangle$, where $p \in \mathbb{Z}$ is prime.
2. All maximal ideals of $\mathbb{Z} * \mathbb{Z}$ are given by $I_3 = \langle(1,-1+p)\rangle$, and $I_4 = \langle(p,1-p)\rangle$, where $p \in \mathbb{Z}$ is prime.
3. The invertible elements of $\mathbb{Z}*\mathbb{Z}$ are $(1,0)$, $(-1,0)$, $(1,-2)$, and $(-1,2)$.
4. The set of zero divisors of $\mathbb{Z}*\mathbb{Z}$ is $S = \{(0,y) \mid y \in \mathbb{Z}\} \cup \{(x,-x) \mid x \in \mathbb{Z}\}$.

Proof. We use the results of Theorem 2.4 and the inverse of the isomorphism φ from Theorem 2.2 as applied in Corollary 2.3. The proof entails following the ideal generators under the mapping $\varphi^{-1}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ given by $\varphi^{-1}(z,r) = (z,-z+r)$.

1. Let $p \in \mathbb{Z}$ be any prime. The prime ideals of $\mathbb{Z} \times \mathbb{Z}$ are of the form $\mathbb{Z} \times p\mathbb{Z} =< (1,p) >$, $p\mathbb{Z} \times \mathbb{Z} =< (p,1) >$, $\{0\} \times \mathbb{Z} =< (0,1) >$, and $\mathbb{Z} \times \{0\} =< (1,0) >$. Applying the mapping φ^{-1} we arrive at de Alwis' list of the prime ideals of $\mathbb{Z}*\mathbb{Z}$: $I_1 = \varphi^{-1}(< (0,1) >) =< (0,1) >$, $I_2 = \varphi^{-1}(< (1,0) >) =< (1,-1) >$, $I_3 = \varphi^{-1}(< (1,p) >) =< (1,-1+p) >$, and $I_4 = \varphi^{-1}(< (p,1) >) =< (p,-p+1) >$.
2. The maximal ideals of $\mathbb{Z}\times\mathbb{Z}$ are of two forms: $\mathbb{Z}\times p\mathbb{Z}$ and $p\mathbb{Z}\times\mathbb{Z}$. Hence, the maximal ideals in $\mathbb{Z} * \mathbb{Z}$ are of the form $I_3 = \varphi^{-1}(< (1,p) >) = < (1,-1+p) >$, and $I_4 = \varphi^{-1}(< (p,1) >) =< (p,-p+1) >$.
3. The only invertible elements in $\mathbb{Z} \times \mathbb{Z}$ are $(1,1)$, $(-1,-1)$, $(1,-1)$, and $(-1,1)$. Therefore, the only invertible elements in $\mathbb{Z}*\mathbb{Z}$ are $\varphi^{-1}(1,1) = (1,0)$, $\varphi^{-1}(-1,-1) = (-1,0)$, $\varphi^{-1}(1,-1) = (1,-2)$, and $\varphi^{-1}(-1,1) = (-1,2)$.
4. All zero divisors in $\mathbb{Z} \times \mathbb{Z}$ have the form $(z,0)$ or $(0,z)$, where z is any integer. Thus, the zero divisors in $\mathbb{Z} * \mathbb{Z}$ are of the form $\varphi^{-1}(z,0) = (z,-z)$ or $\varphi^{-1}(0,z) = (0,z)$.

Hence, we have quickly arrived at the results presented in [1].

Reference

1. T. de Alwis, "The Ideal Structure of $\mathbb{Z} * \mathbb{Z}$," *Missouri J. Math. Sci.*, 6 (1994), 116–123.

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