GENERALIZED PYTHAGOREAN TRIPLES AND PYTHAGOREAN TRIPLE PRESERVING MATRICES

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Abstract. Traditionally, Pythagorean triples (PT) consist of three positive integers, $(x, y, z) \in \mathbb{Z}_+^3$, such that $x^2 + y^2 = z^2$, and Pythagorean triple preserving matrices (PTPM) A are 3×3 matrices with entries in the real numbers $\mathbb{R}$, such that the product $(x, y, z)A$ is also a Pythagorean triple. In this paper, we study PT and PTPM from the view of projective geometry, and extend the results concerning PT and PTPM from integers to any commutative ring with identity. In particular, we use the method of polynomial parametrization for projective conics to obtain the general form of PT over any commutative ring with identity. In addition, we view the PTPM as projective transformations and formulate the general form of a PTPM over any commutative ring with identity.

1. Introduction. A *Pythagorean triple* is a set of three integers x, y, z such that $x^2 + y^2 = z^2$; the triple is said to be *primitive* if $\gcd(x, y, z) = 1$. The best known example of a primitive Pythagorean triple is $(3, 4, 5)$ and the non-primitive Pythagorean triples are $(3k, 4k, 5k)$ where $k \in \mathbb{Z}$, the set of integers. Many undergraduate number theory textbooks introduce Pythagorean triples and provide a characterization of all primitive Pythagorean triples. For example, Theorem 12.1 in [2] states:

Theorem 1.1. All the solutions of the Pythagorean equation $x^2 + y^2 = z^2$; $x, y, z > 0$ satisfying the conditions:

$$\gcd(x, y, z) = 1, \;\; 2|x,$$

are given by the formulas

$$x = 2st, \;\; y = s^2 - t^2, \;\; z = s^2 + t^2$$

for integers $s > t > 0$ such that $\gcd(s, t) = 1$ and $s \not\equiv t \pmod 2$.

The proof of the theorem utilizes properties such as the Fundamental Theorem of Arithmetic in number theory. In this paper, we study Pythagorean triples from the point of view of projective geometry, and extend some results concerning Pythagorean triples from the integers to any commutative ring with identity. First, we introduce the basic concepts of projective geometry such as projective planes, projective lines and projective curves in Section 2. Then in Section 3 we use the techniques of polynomial parametrization for projective conics to obtain the general form of Pythagorean triples, which extends the result in Theorem 1.1 from the integers to any commutative ring with identity. In Section 4, we view the Pythagorean triple preserving matrices A as projective transformations, formulate the general form of A, and extend the results in [3] and [4] to any commutative ring with identity. We provide illustrative examples over different commutative rings with identity in Sections 3 and 4.

2. Projective Conics. In this section, we will introduce the concept of projective conics. First, we introduce the definition of the projective plane. In the Euclidean plane or the Affine plane, we can usually assign a coordinate for the points in the plane. For example, let $\mathbb{K}^2 = \{(x,y) : x, y \in \mathbb{K}\}$, where $\mathbb{K}$ is a field. The construction of the projective plane can be considered as the following. In $\mathbb{K}^3$, an affine plane $\mathbb{K}^2$ is considered as a plane L in $\mathbb{K}^3$ which does not pass through the origin $O = (0,0,0)$. There is a one-to-one correspondence between the points in $\mathbb{K}^2$ and the lines through the origin O in $\mathbb{K}^3$ but not in $\mathbb{K}^2$. For any point $A \in L$, there is a unique line through O and $\overleftrightarrow{AB} \cap L = \{A\}$. We consider the lines through O in the xy-plane as "points at infinity" in $\mathbb{K}^3$.

Definition 2.1. A *point* will be represented by the ordered triple of numbers $(x,y,z) \in \mathbb{K}^3 - \{(0,0,0)\}$, and all triples of the form $a(x,y,z) = (ax, ay, az)$, $a \neq 0$ in $\mathbb{K}$, will represent the same point as (x,y,z) in the projective plane. The projective plane $\mathbb{PK}^2$ is the set of equivalence classes on $\mathbb{K}^3 - \{(0,0,0)\}$,

$$\mathbb{PK}^2 = (\mathbb{K}^3 - \{(0,0,0)\})/\sim,$$

where $(x,y,z) \sim (x', y', z')$ if there is a non-zero $a \in \mathbb{K}$ such that $(x,y,z) = a(x', y', z')$.

Definition 2.2. Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two distinct points. We define the *line* $\overleftrightarrow{AB}$ to be the set of all points $L(x,y,z)$ with the property that

$$x = sx_1 + tx_2, \quad y = sy_1 + ty_2, \quad z = sz_1 + tz_2,$$

for a pair $(s,t) \in \mathbb{PK}$.

Definition 2.3. The *implicit equation of a projective curve* is of the form $F(x,y,z) = 0$, where

$$F(x,y,z) = \sum_{i,j} a_{ij} x^i y^j z^{d-i-j}, \quad a_{ij} \in \mathbb{K}$$

is a homogeneous polynomial in x, y, z of degree d; that is, each monomial in F has total degree d. The *zero set* or the *projective variety* of $F(x,y,z)$ is defined to be

$$\mathbb{V}(F) = \{(x,y,z) \in \mathbb{PK}^2 : F(x,y,z) = 0\}.$$

We call F a *projective curve*.

In this paper, our interest is focused on a special projective curve of degree $d = 2$, and the projective conics of the form $x^2 + y^2 = z^2$. To find all non-trivial Pythagorean triples is equivalent to finding the set of solutions $\{(x, y, z) \in \mathbb{Z}^3 - \{(0, 0, 0)\} : F(x, y, z) = x^2 + y^2 - z^2 = 0\}$. This can be solved by providing a polynomial parametrization for the projective conics.

3. Parametrization of a Conic. Conics can be represented by an implicit equation or by parametric equations. Usually, we use either trigonometric functions or polynomial functions as parametric representations of conics. Studying the Pythagorean triples via polynomial equations is the prime concern of this paper.

Definition 3.1. Let $\mathbb{K}$ be a field. A map $\lambda \colon \mathbb{PK} \to \mathbb{PK}^2$ given by $\lambda(s, t) = (x(s, t), y(s, t), z(s, t))$ provides a parametrization which is a projective plane curve. A *polynomial parametrization* of a projective plane curve F is a set of homogeneous polynomial functions $x(s, t), y(s, t), z(s, t)$ satisfying the following two conditions:

1. for all but finitely many values of $(s, t) \in \mathbb{PK}$, the functions $x(s, t), y(s, t), z(s, t)$ are defined and $F(x(s, t), y(s, t), z(s, t)) = 0$;
2. for any point $(x, y, z) \in \mathbb{PK}^2$ such that $F(x, y, z) = 0$, there is a unique point $(s, t) \in \mathbb{PK}$ such that $x = x(s, t), y = y(s, t), z = z(s, t)$ with a finite number of exceptions.

It should be noted that it is not always possible to find a polynomial parametrization from an implicit algebraic equation. But, projective conics, together with projective straight lines, form the only complete class of projective curves that have polynomial parametrizations. However, not all projective curves with degree greater than or equal to 3 have polynomial parametrization.

The techniques which are used in parametrization of conics are also used in finding integral solutions of polynomial equations in several variables. The subject is named after Diophantus. One of the classical problems is finding Pythagorean triples. We will discuss how to find all Pythagorean triples from the angle of parametrization which is called the *moving line method*. Essentially, we take the geometric approach to find Pythagorean triples, and convert the original problem to the problem of finding a rational parametrization of the projective variety $\mathbb{V}(F) = \mathbb{V}(x^2 + y^2 - z^2)$. This method is not limited to finding integer Pythagorean triples. We introduce an idea called *generalized Pythagorean triples*.

Definition 3.2. Let R be a commutative ring with identity. A *generalized Pythagorean triple* is a set of triples (x, y, z) such that $x, y, z \in R$ and $x^2 + y^2 = z^2$; the triple is said to be *primitive* if $\gcd(x, y, z) = 1$.

Algorithm. Moving Line Method to Find Generalized Pythagorean Triples in a Commutative Ring R with Identity

1. Let $\mathbb{K}$ be the rational field over R. Choose a point $P(0,1,1) \in \mathbb{V}(F(x,y,z)) \subset \mathbb{PK}^2$.
2. Draw a line ℓ through P and any other point $(a,b,c) \in \mathbb{PK}^2 - \mathbb{V}(F)$ such that $a,b,c \in R$ and $a^2+b^2-c^2 \neq 0$. We obtain the parametric equation of ℓ by Definition 2.2
$$x(s,t) = ta, \;\; y(s,t) = s+tb, \;\; z = s+tc,$$
for a pair (s,t) such that not both s,t are zero, and $s,t \in R$.
3. Solve $F(x(s,t),y(s,t),z(s,t)) = 0$ for (s,t) so that s,t can be expressed in terms of (a,b,c), and simplify the expression.

We claim that this procedure implies the following theorem.

Theorem 3.3. All the solutions of the generalized Pythagorean equation $x^2+y^2=z^2$ over the commutative ring R with identity satisfying the condition $\gcd(x,y,z)=1$, are given by one of the following two formulas:
$$x = 2mn, \;\; y = m^2-n^2, \;\; z = m^2+n^2, \;\; \gcd(m,n)=1, \;\; m,n \in R, \quad (1)$$
$$x = m^2-n^2, \;\; y = 2mn, \;\; z = m^2+n^2, \;\; \gcd(m,n)=1, \;\; m,n \in R. \quad (2)$$

Proof. Let $\mathbb{K}$ be the rational field over R. We will follow the procedure and focus on the last step. By the procedure, we have
$$\begin{aligned} F(x(s,t),y(s,t),z(s,t)) &= F(ta, s+tb, s+tc) \\ &= (ta)^2 + (s+tb)^2 - (s+tc)^2 \\ &= t^2(a^2+b^2-c^2) + 2st(b-c) \\ &= t(t(a^2+b^2-c^2) + 2s(b-c)) \\ &= 0. \end{aligned}$$

Note that $t(t(a^2+b^2-c^2)+2s(b-c)) = 0$ implies $t=0$ or $t(a^2+b^2-c^2)+2s(b-c)=0$.

With $t=0$, we obtain the trivial Pythagorean triple $(0,s,s)$ for some non-zero $s \in R$.

If $t(a^2+b^2-c^2)+2s(b-c)=0$, then we have $(s,t) = (a^2+b^2-c^2, 2(c-b))$ and $s,t \in R$. Note since $(a,b,c) \notin \mathbb{V}(F)$, we have $s = a^2+b^2-c^2 \neq 0$. Inserting the above s,t into
$$x(s,t) = ta, \;\; y(s,t) = s+tb, \;\; z = s+tc$$
we obtain
$$x = 2(c-b)a, \;\; y = a^2+b^2-c^2+2(c-b)b, \;\text{ and } z = a^2+b^2-c^2+2(c-b)c,$$

which is equivalent to

$$x = 2(c-b)a, \;\; y = a^2 - (c-b)^2, \;\; z = a^2 + (c-b)^2.$$

To simplify the notation, we will let $m = a$ and $n = c - b$. Since $a, b, c \in R$, we must have that $m, n \in R$, and

$$x = 2mn, \;\; y = m^2 - n^2, \;\; z = m^2 + n^2.$$

In addition, $\gcd(m, n) = 1$, otherwise, $\gcd(x, y, z) \neq 1$.

Note that if we choose $P(1, 0, 1)$ in step one in the algorithm *Moving Line Method to Find Generalized Pythagorean Triples in a Commutative Ring R with Identity*, then we obtain the form

$$x = m^2 - n^2, \;\; y = 2mn, \;\; z = m^2 + n^2, \;\; \gcd(m, n) = 1, \;\; m, n \in R.$$

This is due to the symmetry relation of the Pythagorean triples.

Example 3.4. When $R = \mathbb{Z}_+$, the set of positive integers, we obtain Theorem 1.1 concerning integer Pythagorean triples. If we set $m = 3$ and $n = 2$, then $(12, 5, 13)$ is a Pythagorean triple.

When $R = \mathbb{Z}[i]$, the Gaussian integers, if we set $m = 3+i$ and $n = 2+i$, then $(10 + 10i, 5 + 2i, 11 + 10i)$ is a Pythagorean triple.

When $R = \mathbb{Z}[x, y, z]$, the polynomial ring of 3-variables, if we let $m = x+y$ and $n = z$, then $(2z(x+y), (x+y)^2 - z^2, (x+y)^2 + z^2)$ is a Pythagorean triple.

When $R = \mathbb{Z}[x]$, if we let $m = ax^k, n = b$ with $a, b \in \mathbb{Z}$, $k \in \mathbb{N}$, and $\gcd(m, n) = 1$ then $(2abx^k, a^2x^{2k} - b^2, a^2x^{2k} + b^2)$ gives a class of primitive Pythagorean triples.

When $R = \mathbb{Z}_p$ for $p \geq 3$ and $p \in \mathbb{N}$, the ring of integers mod p, if we set $a = 2mn, b = m^2 - n^2$ and $c = m^2 + n^2$ with $m, n \in \mathbb{Z}$, then (a, b, c) is a Pythagorean triple in $\mathbb{Z}_p$. Moreover (a, b, c) is a Pythagorean triple if and only if $(p - a, p - b, p - c)$ is a Pythagorean triple.

Let $R^* = M_n(R)$ be the ring of all $n \times n$ matrices with entries in the commutative ring R with identity. Then the center of ring R^* is the set of all diagonal matrices in R^*, which is a commutative ring with identity. If $D_1, D_2 \in R^*$, then $(2D_1D_2, D_1^2 - D_2^2, D_1^2 + D_2^2)$ is a Pythagorean triple in R^*.

4. Pythagorean Triple Preserving Matrix.

Definition 4.1. An invertible 3×3 matrix $A = [a_{ij}]$ is called a *Pythagorean triple preserving matrix of type I (or, II)* if A maps a Pythagorean triple of the form of Equation (1) (or, Equation (2)) to a Pythagorean triple of the form Equation (1) (or, Equation (2)), that is, if $\mathbf{a}$ is a Pythagorean triple of the form of Equation (1) (or, (2)), then $\mathbf{a}A = \mathbf{b}$ is also a Pythagorean triple of the form of Equation (1) (or, (2)).

Similarly, we call an invertible 3×3 matrix $A = [a_{ij}]$ a *Pythagorean triple preserving matrix of type I - II (or, II-I)* if A maps a Pythagorean triple of the form of Equation (1) (or, Equation (2)) to a Pythagorean triple of the form Equation (2) (or, Equation (1)), that is, if $\mathbf{a}$ is a Pythagorean triple of the form of Equation (1) (or, (2)), then $\mathbf{a}A = \mathbf{b}$ is also a Pythagorean triple, and $\mathbf{b}$ is of the form of Equation (2) (or, (1)).

We call an invertible 3×3 matrix $A = [a_{ij}]$ a *Pythagorean triple preserving matrix* if matrix A is a Pythagorean triple preserving matrix of type I, type II, type I-II, and II-I. That is, if $\mathbf{a}$ is a Pythagorean triple of either form, then $\mathbf{a}A = \mathbf{b}$ is also a Pythagorean triple of either form.

From the view of projective geometry, we can relate this Pythagorean preserving matrix to the projective map.

Definition 4.2. Let $\mathbb{K}$ be a field, a *projective transformation*, or *projective equivalence*, of a projective space is a map $T\colon \mathbb{PK}^n \to \mathbb{PK}^n$ of the form

$$T(\mathbf{p}) = \mathbf{p}A_n, \text{ for all } \mathbf{p} \in \mathbb{PK}^n,$$

where A_n is represented by an invertible $(n+1) \times (n+1)$ matrix over $\mathbb{K}$, the ring of reals. If $n = 1$, then the map A_n is of size 2×2, a projective transformation of the projective line. If $n = 2$, then the map A_n is of size 3×3, a projective transformation of the projective plane.

Since the Pythagorean triples are the solutions for the equation $x^2 + y^2 - z^2 = 0$, we may write this as a matrix form

$$[\,x, y, z\,]\, C\, [\,x, y, z\,]^T = 0, \text{ where } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We may consider a Pythagorean preserving matrix as a projective transformation which sends a Pythagorean triple (x, y, z) to another Pythagorean triple (X, Y, Z).

Lemma 4.3. If A is a Pythagorean preserving matrix (that is A is a Pythagorean preserving matrix of all the types), then $ACA^T = C$. Moreover, $\det(A) = \pm 1$.

Proof. Let A be a Pythagorean preserving matrix which maps Pythagorean triples (x, y, z) to Pythagorean triples (X, Y, Z). Then

$$\begin{aligned} X^2 + Y^2 - Z^2 &= [X, Y, Z]C[X, Y, Z]^T \\ &= [x, y, z]ACA^T[x, y, z]^T \\ &= [x, y, z](ACA^T)[x, y, z]^T \\ &= 0. \end{aligned}$$

The fact that (x, y, z) is also a Pythagorean triple implies that $ACA^T = C$, and $\det(ACA^T) = \det(C)$ implies that $\det(A) = \pm 1$.

Below, we state and prove a theorem concerning some properties of a Pythagorean triple preserving matrix A. A result similar to this theorem and much more regarding the Pythagorean triple preserving matrix can be found in [1]. Moreover, in the paper [5], the matrix A is obtained from a different perspective. We choose to use the terminology Type a-b (where a and b are either I or II) to cover all cases.

Theorem 4.4. Let R be any commutative ring with identity and let $\mathbb{K} = \mathbb{Q}[R]$ be the ring of polynomials in R with coefficients in $\mathbb{Q}$ (that is, if $f \in \mathbb{K}$, then $f = \sum_{i=1}^{n} a_i(r_1^{i_1} \cdots r_j^{i_j})$ for some $a_i \in \mathbb{Q}$ and some $r_1, \ldots, r_j \in R$). Let (x, y, z) be any generalized Pythagorean triple over R of the form of Equation (1). A matrix A is a Pythagorean preserving matrix of type I with entries in $\mathbb{K}$ if and only if

$$A = \begin{bmatrix} ru + st & rt - su & rt + su \\ rs - tu & \dfrac{(r^2 - s^2) - (t^2 - u^2)}{2} & \dfrac{(r^2 + s^2) - (t^2 + u^2)}{2} \\ rs + tu & \dfrac{(r^2 - s^2) + (t^2 - u^2)}{2} & \dfrac{(r^2 + s^2) + (t^2 + u^2)}{2} \end{bmatrix},$$

where $r, s, t, u \in R$ and $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$ is a projective transformation from $\mathbb{PK}$ to $\mathbb{PK}$ such that $(m, n)\begin{bmatrix} r & s \\ t & u \end{bmatrix} = (M, N)$.

Proof. $(\Rightarrow)$ First, let A be a Pythagorean triple preserving matrix with entries in $\mathbb{K}$. We will show that A is of the given form.

Relating this with our parametrization, we know that there is a correspondence between the projective transformations on the projective line and projective planes, respectively. Therefore, we have the following one-to-one correspondence:

$$\begin{gathered} (\mathbb{PK}|_{\mathbb{P}R} \to \mathbb{PK}|_{\mathbb{P}R}) \to (\mathbb{PK}^2|_{\mathbb{P}R^2} \to \mathbb{PK}^2|_{\mathbb{P}R^2}), \\ ((m, n) \to (M, N)) \to ((x, y, z) \to (X, Y, Z)), \\ ((m, n)A_1 = (M, N)) \to ((x, y, z)A = (X, Y, Z)), \\ ((m, n)\begin{bmatrix} r & s \\ t & u \end{bmatrix} = (M, N)) \to ((x, y, z)A = (X, Y, Z)), \end{gathered}$$

where A is the Pythagorean preserving matrix, and A_1, the projective line transformation, is a 2×2 matrix $\begin{bmatrix} r & s \\ t & u \end{bmatrix}$, such that $r, s, t, u \in R$ and

$(m,n)\begin{bmatrix} r & s \\ t & u \end{bmatrix} = (M,N)$. This implies the following equations:

$$\begin{aligned}
M^2 &= m^2r^2 + 2mnrt + n^2t^2,\\
N^2 &= m^2s^2 + 2mnsu + n^2u^2,\\
2MN &= 2m^2rs + 2mn(ru+st) + 2n^2tu,\\
M^2 - N^2 &= m^2(r^2-s^2) + 2mn(rt-su) + n^2(t^2-u^2),\\
M^2 + N^2 &= m^2(r^2+s^2) + 2mn(rt+su) + n^2(t^2+u^2).
\end{aligned}$$

When A is a Pythagorean triple preserving matrix of type I, we have the following relations

$$\begin{aligned}
(x,y,z)A &= (X,Y,Z),\\
(2mn, m^2-n^2, m^2+n^2)A &= (2MN, M^2-N^2, M^2+N^2),\\
(2mn, m^2, n^2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} A &= \\
&(2mn, m^2, n^2)\begin{bmatrix} ru+st & rt-su & rt+su \\ 2rs & r^2-s^2 & r^2+s^2 \\ 2tu & t^2-u^2 & t^2+u^2 \end{bmatrix},
\end{aligned}$$

which implies

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} ru+st & rt-su & rt+su \\ 2rs & r^2-s^2 & r^2+s^2 \\ 2tu & t^2-u^2 & t^2+u^2 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} ru+st & rt-su & rt+su \\ 2rs & r^2-s^2 & r^2+s^2 \\ 2tu & t^2-u^2 & t^2+u^2 \end{bmatrix}\\
&= \begin{bmatrix} ru+st & rt-su & rt+su \\ rs-tu & \dfrac{(r^2-s^2)-(t^2-u^2)}{2} & \dfrac{(r^2+s^2)-(t^2+u^2)}{2} \\ rs+tu & \dfrac{(r^2-s^2)+(t^2-u^2)}{2} & \dfrac{(r^2+s^2)+(t^2+u^2)}{2} \end{bmatrix}.
\end{aligned}$$

($\Leftarrow$) On the other hand, if A is of the given form, let $(x,y,z) \in R$ be a generalized Pythagorean triple. By Theorem 3.3, $(x,y,z) = (2mn, m^2 -$

$n^2, m^2+n^2)$ with $\gcd(m,n)=1$ and $m,n\in R$. Thus,

$$\begin{aligned}
(x,y,z)M &= [2mn, m^2-n^2, m^2+n^2]A \\
&= [2mn, m^2-n^2, m^2+n^2] \\
&\times \begin{bmatrix} ru+st & rt-su & rt+su \\ rs-tu & \dfrac{(r^2-s^2)-(t^2-u^2)}{2} & \dfrac{(r^2+s^2)-(t^2+u^2)}{2} \\ rs+tu & \dfrac{(r^2-s^2)+(t^2-u^2)}{2} & \dfrac{(r^2+s^2)+(t^2+u^2)}{2} \end{bmatrix} \\
&= [2(mr+nt)(ms+nu),\ \ (mr+nt)^2-(ms+nu)^2, \\
&\qquad (mr+nt)^2+(ms+nu)^2] \\
&= [2MN,\ \ M^2-N^2,\ \ M^2+N^2],
\end{aligned}$$

where $r,s,t,u\in R$ such that $(m,n)\begin{bmatrix} r & s \\ t & u\end{bmatrix} = (M,N)$.

Remark 4.5. By appropriate change of rows or columns of a Pythagorean triple preserving matrix of type I, we will obtain all other types of Pythagorean triple preserving matrices.

Example 4.6. Let $R=\mathbb{Z}$ be the ring of integers. Let

$$\begin{bmatrix} r & s \\ t & u\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1\end{bmatrix}.$$

The Pythagorean triple preserving matrix of type I-I with entries in $\mathbb{Q}$, the rational numbers, is

$$A = \begin{bmatrix} -1 & 1/2 & 1 \\ 1 & 1/2 & -1/2 \\ -1 & 1/2 & 3/2 \end{bmatrix}.$$

We check that

$$(4,3,5)A = (-6,8,10).$$

Example 4.7. Let $R=\mathbb{Z}[x]$ be the polynomial rings with real coefficients. Let

$$\begin{bmatrix} r & s \\ t & u\end{bmatrix} = \begin{bmatrix} x & x-1 \\ x+1 & x\end{bmatrix}.$$

The Pythagorean triple preserving matrix of type I-I is

$$A = \begin{bmatrix} 2x^2-1 & 2x & 2x^2 \\ -2x & -1 & -2x \\ 2x^2 & 2x & 2x^2+1 \end{bmatrix}.$$

If we let $(m,n)=(2,1)$, then

$$(M,N)=(2,1)\begin{bmatrix} r & s \\ t & u \end{bmatrix}=(2,1)\begin{bmatrix} x & x-1 \\ x+1 & x \end{bmatrix}=(3x+1,3x-2).$$

We have two Pythagorean triples:

$$(x,y,z)=(4,3,5),\ \ (X,Y,Z)=(18x^2-6x-4,18x-3,18x^2-6x+5).$$

We check that

$$(x,y,z)A=(X,Y,Z).$$

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References

1. P. Braza, J. Tong, and M. Zhan, "Linear Transformations on Pythagorean Triples," *International Journal of Mathematical Education in Science and Technology*, 35 (2004), 755–762.
2. D. M. Burton, *Elementary Number Theory*, 6th edition, McGraw-Hill Higher Education, New York, 2007.
3. L. Palmer, M. Ahuja, and M. Tikoo, "Finding Pythagorean Triple Preserving Matrix," *Missouri Journal of Mathematical Sciences*, 10 (1998), 99–105.
4. L. Palmer, M. Ahuja, and M. Tikoo, "Constructing Pythagorean Triple Preserving Matrices," *Missouri Journal of Mathematical Sciences*, 10, (1998), 159–168.
5. J. Tong, "Conjugates of Pythagorean Triples," *Mathematical Gazette*, 87, (2003), 496–498.

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