THE ALGEBRAIC CASE OF A CONJECTURE OF NAKANISHI

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Abstract. In this paper we provide a proof of the algebraic case of a conjecture of Nakanishi concerning a proposed unknotting operation. Specifically, we show, using only basic knot theory techniques, that any algebraic knot or link can be unknotted by a sequence of so-called (2,2)-moves.

1. Introduction. The essence of knot theory lies in the search for properties that can be used to distinguish knots and links. These properties measure the inherent structure of a knot without regard to how the knot is embedded in space. Also of interest is the study of actions that can unknot a given knot. That is, operations on a knot which, instead of preserving the structure, transform it into a trivial knot or link. The simplest of these actions is the crossing change, shown in Figure 1.

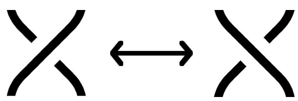


Figure 1. The crossing change.

We leave it as an exercise to the reader to show that by changing some crossings in a knot projection we can transform it into a projection of the trivial knot. It so happens that some knots may be unknotted by changing only one crossing, while others require two or more crossing changes. Hence, the crossing change gives us not only a method for unknotting knots, but also a property that can distinguish between knots. By counting the minimum number of crossing changes needed to unknot a particular knot we find the **unknotting number** of the knot, which gives us a basic (though not necessarily easy to calculate) measure of the complexity of a knot. In addition to the crossing change there are several other known operations that will unknot a given knot. For a partial list see reference [4].

2. Nakanishi's Conjecture. In reference [4], Nakanishi posed the question of whether the moves shown in Figure 2, so-called $(2,2)$-moves, combined with Reidemeister moves, could transform any knot or link into a trivial link.

Figure 2. $(2,2)$-moves.

We can think of this move as cutting out a piece of a knot with either 2 positive or negative crossings and reattaching it after a $90°$ rotation. Note that we call a crossing positive if the overstrand has a positive slope and negative if the overstrand has a negative slope.

In general Nakanishi's conjecture is false, as proved by Przytycki and Dabkowski in [5]. However, it is true that every algebraic knot or link can be unknotted by a sequence of $(2,2)$- and Reidemeister moves. Proving this proposition is the main goal of this paper.

2.1 Notation and Preliminaries. Hereafter we will refer to the piece of a knot on which we perform the $(2,2)$-move as a **bigon**. Adams defines a **tangle** in a knot or link as "*a region in the projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times.*" [1] We use the phrase **integer tangle** to denote a section of half twists that is equivalent to a single integer in the Conway notation of a knot. For an integer m, an **m-tangle** is an integer tangle with $|m|$ crossings, all of which are positive or negative. Throughout, the default configuration of a single integer tangle will be a horizontal row of crossings as shown in Figure 3. Hence, an integer tangle is a tangle, but not all tangles are integer tangles.

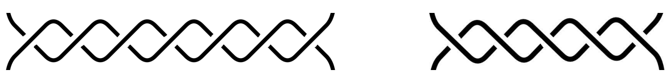


Figure 3. A 7-tangle and a (-5)-tangle.

Finally, since Reidemeister moves are the standard moves on a knot that do not alter the structure of the knot, we will always allow for the possibility of a sequence of Reidemeister moves between $(2,2)$-moves whether or not we explicitly state so. Hence, we generally omit mention of these moves in the remainder of the paper unless we are speaking of a specific move.

2.2 Equivalence. We say two knots or links are $\mathbf{(2,2)}$**-equivalent** if one can be transformed into the other by a sequence of $(2,2)$-moves and Reidemeister moves. Notice that this gives rise to an equivalence relation on knots and links. We state this formally in the following proposition and leave the proof as an exercise.

Proposition 2.1. Nakanishi's $(2,2)$-move defines an equivalence relation on knots and links.

Given this proposition, Nakanishi's conjecture can be restated as: "Does every link belong to some $(2,2)$-equivalence class containing a trivial link of some number of components?" This result has been established in [2], but we present a different approach to the proof that relies only on basic combinatorial knot theory techniques.

3. The $(\mathbf{2},\mathbf{2})$-Move on an Integer Tangle. By fixing the endpoints of an integer tangle with standard labeling (NW, NE, SW and SE), we can define the $(2,2)$-equivalence classes of integer tangles analogous to the $(2,2)$-equivalence classes for links in Proposition 2.1. We fix the endpoints in place to trap the crossings particular tangle so that the ends cannot twist and undo any of the crossings. Once we have these equivalence classes, we will define the term algebraic and consider the action of a $(2,2)$-move on an algebraic knot or link. We note here that the $(2,2)$-equivalence class of an integer tangle may contain non-integer tangles. However, our primary interest is in which integer tangles are $(2,2)$-equivalent so this will not pose a problem. The following proposition will allow us to define these equivalence classes.

Proposition 3.1. Any integer tangle with five or more crossings is $(2,2)$-equivalent to an integer tangle with five fewer crossings.

Proof. Let M be an integer tangle with five or more crossings and fixed endpoints. Perform a $(2,2)$-move on the leftmost bigon. The resulting tangle is no longer an integer tangle, but it can be deformed by Reidemeister moves into a new tangle with one fewer crossing. A subsequent $(2,2)$-move on the new leftmost bigon will produce a new tangle which will admit a pair of Type II Reidemeister moves. These moves reduce the number of crossings by 4 and result in an integer tangle with five fewer crossings.

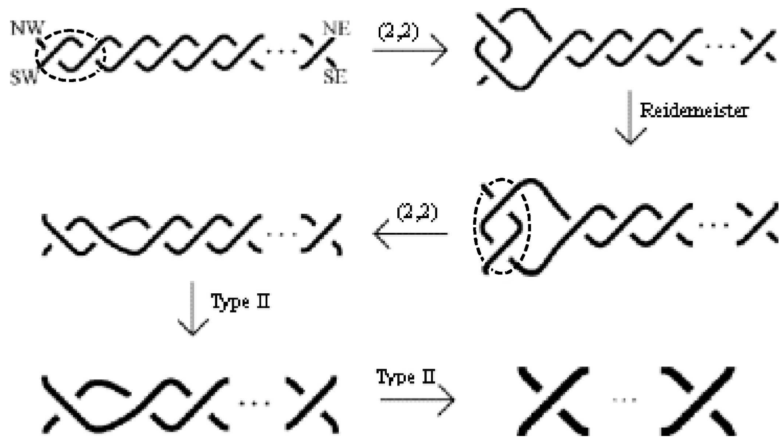
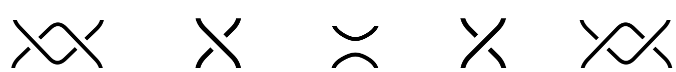


Figure 4. Transformation of an m-tangle into an $(m-5)$-tangle.

Of course, once we can subtract five crossings we can do it as many times as necessary, as long as there are at least 5 crossings. It follows that every integer tangle is $(2,2)$-equivalent to a k-tangle with $|k| \leq 4$. But we can actually do better. For example, if we have a 3-tangle, then we can apply the method from the proof of Proposition 3.1, to get an integer tangle with 2 negative crossings. To see this, imagine applying the method of the previous proof, but before the first $(2,2)$-move, do two Type II Reidemeister moves to create five positive crossings with two adjacent negative crossings. Then the procedure will remove the five positive crossings and leave us with the two negative crossings that the Reidemeister moves introduced. Similarly, we can apply the same method to transform a (-4)-tangle into a 1-tangle. It follows that we can reduce the number of crossings in an integer tangle to 2 or less. We state this formally as a corollary.

Corollary 3.1. Every m-tangle is $(2,2)$-equivalent to a k-tangle for some $k \in \{-2,-1,0,1,2\}$.

Thus we may partition the set of integer tangles into five equivalence classes under the $(2,2)$-equivalence relation. We choose the following tangles as our representatives and call them **reduced integer tangles**.



-2-tangle -1-tangle 0-tangle 1-tangle 2-tangle

Figure 5. Reduced integer tangles.

4. Tangle Addition and Multiplication Under $(\mathbf{2},\mathbf{2})$-moves A **rational tangle** is a tangle that is made up of integer tangles that have been combined by some sequence of additions and multiplications. In this section we give an overview of these operations and see what happens to these new tangles under $(2,2)$-equivalence.

4.1 Tangle Addition. The addition of two tangles is defined by taking the two tangles side by side and connecting the NE endpoint of the left tangle to the NW endpoint of the right tangle and the SE endpoint of the left tangle to the SW endpoint of the right tangle. Here we focus only on the addition of integer tangles, but as a consequence of later results, we can extend our actions to any rational tangle. Figure 6 shows the addition of two integer tangles. Notice that if we add a k-tangle to a j-tangle, then we have a (possibly non-integer) horizontal tangle with $|k|+|j|$ crossings, some of which may be removed using Type II Reidemeister moves to obtain an integer tangle with $|k+j|$ positive or negative crossings which is subsequently $(2,2)$-equivalent to one of our reduced integer tangles by Corollary 3.1. Thus, the sum of two integer tangles is $(2,2)$-equivalent to one of the five reduced integer tangles in Figure 5. One nice feature of this is that the set of integer tangles under addition and $(2,2)$-equivalence forms a group which is isomorphic to integer addition mod 5.

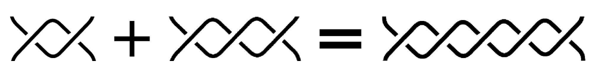


Figure 6. The sum of two tangles.

4.2 Tangle Multiplication. The product of two tangles is defined by reflecting the left tangle across a NW–SE mirror and then adding it to the right tangle, as shown in Figure 7. Since we must reflect the tangle before we add, we note that the five reduced integer tangles will not suffice when discussing tangle multiplication. In addition to these, we need to include the reflection of the zero tangle, called the infinity tangle, shown in Figure 8. Reflection of any of the other four tangles produces another one of these tangles, after a possible $(2,2)$-move.

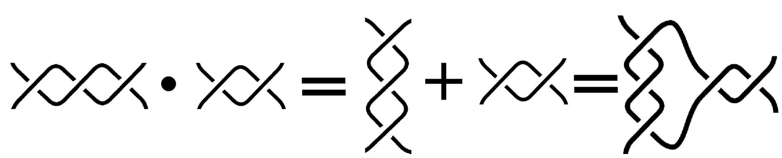


Figure 7. The product of two tangles.

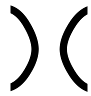
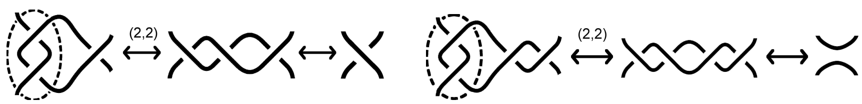


Figure 8. The infinity tangle.

By noting that a $-m$-tangle is just the mirror image of an m-tangle and observing that as we reduce a tangle its mirror image is reduced by the opposite $(2,2)$-move, we reduce the number of cases to check effectively by half. Now, we know that all integer tangles are $(2,2)$-equivalent to an integer tangle of 2 or fewer crossings. Moreover, in tangle multiplication we are only concerned with cases in which two reduced integer tangles, or the infinity tangle, are multiplied and result in a tangle with more than two crossings. Two of these tangle products and their respective reductions are shown in Figure 9.



$2 \cdot 1$ $2 \cdot 2$

Figure 9. Two cases for integer tangle multiplication.

We note that these two examples, along with their mirror images, represent the entirety of tangle products of two reduced integer tangles, or the infinity tangle, for $(2,2)$-equivalence that have crossing numbers greater than two. (Notice that $2 \cdot -2$ is isotopic to $-2 \cdot -1$.) This means that the product of any two integer tangles or the infinity tangle can be reduced to a tangle of two or fewer crossings with the possible inclusion of some trivial components. Inductively, we see that any finite number of tangles can be multiplied together and reduced to a tangle with two or fewer crossings and possibly one or more uncrossed loops.

5. Algebraic Knots and Links Under the $(\mathbf{2},\mathbf{2})$-Move. An **algebraic tangle** is a tangle that results from from sequence of multiplications and additions of rational tangles, and an **algebraic link** is defined as the closure of an algebraic tangle. This closure comes from connecting the NW endpoint of a rational tangle to its NE endpoint and its SW endpoint to its SE endpoint to form a closed loop, or a set of closed loops. We now have all the machinery in place to proceed to our main theorem.

Theorem 5.1. Every algebraic link is $(2,2)$-equivalent to a trivial link.

Proof. Since the $(2,2)$-move reduces any sum or product of integer tangles to an integer tangle of two or fewer crossings, we can systematically simplify any algebraic link so that we have two or fewer crossings. Hence, any algebraic link is $(2,2)$-equivalent to a link of two or fewer crossings. Since all knots of fewer than three crossings are trivial we are left with only links of two or fewer crossings to check. The only nontrivial link that satisfies this condition, up to the inclusion of extra trivial components, is the Hopf link. The reduction of this link is shown in Figure 10. Therefore, every algebraic knot or link is $(2,2)$-equivalent to a trivial knot or link.

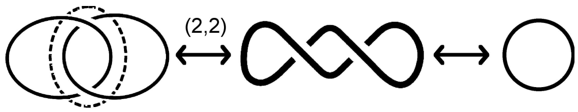


Figure 10. Reduction of the Hopf Link.

6. Counting Components. One final question regarding the number of components remains. It is an easy exercise to show that the figure-eight knot is $(2,2)$-equivalent to a trivial two component link, and we have seen that the Hopf link is $(2,2)$-equivalent to the trivial knot. Thus, it is possible that a knot can be transformed into a trivial link and a link can be transformed into the trivial knot by $(2,2)$-moves. Hence, the last piece of the puzzle is to figure out how the $(2,2)$-move affects the number of components. We can solve this puzzle using colorability. Figure 11 shows that the $(2,2)$-move preserves 5-colorability.

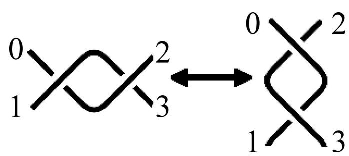


Figure 11. The $(2,2)$-move preserves 5-colorability.

Therefore, since the unknot is not 5-colorable and trivial links are 5-colorable, we can see that the 5-colorable, algebraic knots and links are unknotted into a trivial link and the non-5-colorable, algebraic knots and links are unknotted into the trivial knot. So now the question becomes

> *If a 5-colorable knot or link is transformed into a trivial link by a sequence of* $(2,2)$*-moves, how many components would the link have?*

The answer lies in a matrix associated with coloring. Note that since the $(2,2)$-move preserves 5-colorability, it also preserves the mod 5 rank of the knot. [3] Hence, the number of free variables in the solution to the system of equations corresponds to the number of extra loops we are able to pull off as we perform our sequence of $(2,2)$-moves. Counting the final uncrossed loop gives us the number of components of the trivial link. We state this formally as a theorem whose proof follows from the previous discussion.

Theorem 6.1. If K is a 5-colorable algebraic knot with rank n, then it is $(2,2)$-equivalent to a trivial link with $n+1$ components.

7. Conclusion. We have shown that the family of $(2,2)$-moves is an unknotting operation on the class of algebraic knots and links. Moreover, we know that non-5-colorable algebraic knots and links are unknotted into the trivial knot and that 5-colorable algebraic knots and links are unknotted into a trivial link whose number of components depends on the mod 5 rank of the knot or link. One question that remains is whether we can show that Nakanishi's conjecture is false using elementary methods, rather than relying on the advanced algebraic methods employed by Przytycki and Dabkowski.

Acknowledgements. The authors gratefully acknowledge the helpful comments of the referee. We are particularly grateful for a suggestion that led to a stronger result.

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Mathematics Subject Classification (2000): 57M25, 57-02

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