CC-VERSION OF HERON'S FORMULA

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Abstract. The idea of CC-metric is introduced by Krause [4] and improved by Chen [1]. Later, CC-analogues of some of the topics that include the concept of CC-distance have been studied. In this work, we give the Chinese Checker version of Heron's Formula.

1. Introduction. As it has been stated in [7], Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). Here, the linear structure is the same as the Euclidean one but distance is not *uniform* in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set. Therefore, although the parallel axiom is considered to be valid, Pythagoras' Theorem is not.

The CC-plane geometry is a Minkowski geometry of dimension two with the distance function

$$d_c(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2}-1)\min\{|x_1 - x_2|, |y_1 - y_2|\},$$

where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. That is, the Chinese Checker plane $\mathbb{R}^2_c$ is almost the same as the Euclidean analytical plane $\mathbb{R}^2$. The points and lines are the same, and angles are measured in the same way. However, the distance function is different. According to the definition of d_c-distance the shortest path between the points P_1 and P_2 is the union of a vertical or horizontal line segment and a line segment with the slope 1 or -1.

2. Preliminaries. The following propositions and corollaries give some results of $\mathbb{R}^2_c$.

Proposition 2.1. Let l be a line through the points P_1 and P_2 in the analytical plane. If l has slope m, then

$$d_c(P_1, P_2) = \frac{M}{\sqrt{1+m^2}} d_E(P_1, P_2),$$

where

$$M = \begin{cases} 1 + (\sqrt{2}-1)|m|, & \text{if } |m| \le 1 \\ (\sqrt{2}-1) + |m|, & \text{if } |m| \ge 1. \end{cases}$$

Corollary 2.2. Let P_1, P_2, and X be any three collinear points in $\mathbb{R}^2$. Then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_c(P_1, X) = d_c(P_2, X)$.

Corollary 2.3. If P_1, P_2, and X are any distinct three collinear points in the real plane, then

$$\frac{d_c(P_1,X)}{d_c(P_2,X)} = \frac{d_E(P_1,X)}{d_E(P_2,X)}.$$

Proofs of the above assertions are given in [2]. One of the basic problems in geometric investigations for a given space S with a metric d is to describe the group G of isometries. The following propositions about isometry of CC-plane are given in [2].

Proposition 2.4. Every Euclidean translation is an isometry of $\mathbb{R}_c^2$.

Proposition 2.5. A reflection about the line $y = mx$ in $\mathbb{R}_c^2$ is an isometry if and only if $m \in \{0, \mp 1, \mp(\sqrt{2}-1), \mp(\sqrt{2}+1), \infty\}$.

Proposition 2.6. There are only eight Euclidean rotations by the d_c-distances. In fact, the set of isometric rotations in $\mathbb{R}_c^2$ is

$$R_c = \left\{ r_\theta \mid \theta = k\frac{\pi}{4}, \quad k = 0, 1, \ldots, 7 \right\}.$$

In [2], it is shown that the group of isometries of the plane with respect to the Chinese Checker metric is the semi-direct product of the Dihedral group D_8 and $T(2)$, where D_8 is the (Euclidean) symmetry group of the regular octahedron and $T(2)$ is the group of all translations of the plane.

We need the following definitions given in [6] and [3].

Let ABC be any triangle in the CC-plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line l is called a *base line* of ABC if and only if

1) l passes through a vertex,
2) l is parallel to a coordinate axis,
3) l intersects the opposite side (as a line segment) to the vertex in Condition 1. Clearly, at least one of the vertices of a triangle always has one or two base lines. Such a vertex of a triangle is called a *basic vertex.* A *base segment* is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Finally, we classify the lines of the CC-plane in the following way. Lines with slope $|m| > 1$, $|m| < 1$ or $|m| = 1$ are called a *steep lines*, *gradual lines*, or *separators*, respectively.

3. CC-version of Heron's Formula. Clearly a well-known formula

$$\text{area of a triangle} = (\text{base} \times \text{height})/2$$

is not valid in $\mathbb{R}_c^2$. It is valid if and only if the base is parallel to any one of the coordinate axes or one of the lines $y = \mp x$. In this case, Euclidean and CC-lengths of the base and height are the same. Also one can compute area of a triangle by using the three sides of the triangle. Let the sides of a triangle have lengths a, b, and c, and the semiperimeter $p = (a+b+c)/2$. If $\mathcal{A}$ denote the area, then

$$\mathcal{A}^2 = p(p-a)(p-b)(p-c)$$

is known as Heron's Formula.

Let the sides of a triangle ABC, in the CC-plane, have lengths $\mathbf{a} = d_c(B,C)$, $\mathbf{b} = d_c(A,C)$, and $\mathbf{c} = d_c(A,B)$, and the Chinese Checker perimeter $\mathbf{p} = (\mathbf{a}+\mathbf{b}+\mathbf{c})/2$. The following two propositions give the CC-version of Heron's formula in some special cases.

Proposition 3.1. If one side of a triangle ABC, say BC, is parallel to one of the coordinate axes or lines $y = \mp x$ and none of the angles B and C is an obtuse angle, then for the area $\mathcal{A}$ of $\triangle ABC$

$$\mathcal{A} = \begin{cases} \dfrac{\mathbf{a}}{2(\sqrt{2}-1)}(\mathbf{p}-\mathbf{a}), & \text{if } C1 \text{ is satisfied} \\ \dfrac{\mathbf{a}}{2\sqrt{2}}\left(\mathbf{p}-\mathbf{a}+\dfrac{\mathbf{b}}{\sqrt{2}}\right), & \text{if } C2 \text{ is satisfied} \\ \dfrac{\mathbf{a}}{2\sqrt{2}}\left(\mathbf{p}-\mathbf{a}+\dfrac{\mathbf{c}}{\sqrt{2}}\right), & \text{if } C3 \text{ is satisfied} \\ \dfrac{\mathbf{a}}{2}\left(\mathbf{p}-\dfrac{\mathbf{a}}{\sqrt{2}}\right), & \text{if } C4 \text{ is satisfied,} \end{cases}$$

where

$C1$: BC is parallel to the x-axis (y-axis) and sides AB, AC are on gradual (steep) lines or BC is parallel to one of the lines $y = \mp x$ and one of AB, AC is on the gradual and the others is on the steep line.

$C2$: BC is parallel to the x-axis (y-axis), AB is a gradual (steep) line and AC is a steep (gradual) line or BC is parallel to one of the lines $y = \mp x$ and AB is a steep line and AC is a gradual line.

$C3$: BC is parallel to the x-axis (y-axis), AB is a steep (gradual) line and AC is a gradual (steep) line or BC is parallel to one of the lines $y = \mp x$ and AB is a steep line and AC is a gradual line.

$C4$: BC is parallel to the x-axis (y-axis) and sides AB, AC are on steep (gradual) lines or BC is parallel to one of the lines $y = \mp x$ and AB is a steep line and AC is a gradual line.

Proof. Consider the triangle ABC such that BC is parallel to the x-axis. Let $\mathbf{h} = d_c(A, A')$ and $\mathbf{a}' = d_c(B, A')$, where A' denotes the foot of the altitude from A. According to positions of AB and AC sides, there are four possible cases.

Case I. Let angles B and C be smaller than $\pi/4$. That is, sides AB and AC are on gradual lines as shown in Figure 1.

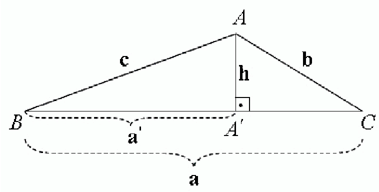

Figure 1.

For the triangles ABA' and $AA'C$ we get

$$\mathbf{c} = \mathbf{a}' + (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{a}' = \mathbf{c} - (\sqrt{2}-1)\mathbf{h}$$

and

$$\mathbf{b} = (\mathbf{a}-\mathbf{a}') + (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{h} = (-\mathbf{a}+\mathbf{b}+\mathbf{c})/2(\sqrt{2}-1),$$

respectively. From these equalities,

$$\mathcal{A} = \frac{1}{2}\mathbf{a}\mathbf{h} = \frac{1}{2}\mathbf{a}\left(\frac{-\mathbf{a}+\mathbf{b}+\mathbf{c}}{2(\sqrt{2}-1)}\right) = \frac{\mathbf{a}}{2(\sqrt{2}-1)}(\mathbf{p}-\mathbf{a}).$$

The statement is also valid if BC is parallel to the y-axis and AB and AC are steep lines since the rotation with the angle $\pi/2$ is an isometry of the CC-plane [2].

Similarly, the statement is valid if BC is parallel to one of the lines $y = \mp x$ and one of AB and AC is on gradual and the others is on a steep line since the rotation with the angle $\pi/4$ is an isometry of the CC-plane [2].

Case II. Assume that side AB is on a gradual line and side AC is on a steep line as shown in Figure 2.

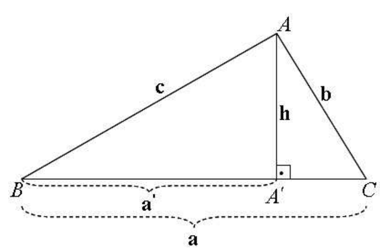

Figure 2.

Using the triangles ABA' and $AA'C$, we obtain

$$\mathbf{c} = \mathbf{a}' + (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{a}' = \mathbf{c} - (\sqrt{2}-1)\mathbf{h}$$

and

$$\mathbf{b} = \mathbf{h} + (\sqrt{2}-1)(\mathbf{a}-\mathbf{a}') \Rightarrow \mathbf{h} = (-\mathbf{a} + (\sqrt{2}+1)\mathbf{b} + \mathbf{c})/2\sqrt{2},$$

respectively. From these equalities,

$$\mathcal{A} = \frac{1}{2}\mathbf{ah} = \frac{1}{2}\mathbf{a}\left(\frac{-\mathbf{a}+(\sqrt{2}+1)\mathbf{b}+\mathbf{c}}{2\sqrt{2}}\right) = \frac{\mathbf{a}}{2\sqrt{2}}\left(\mathbf{p}-\mathbf{a}+\frac{\mathbf{b}}{\sqrt{2}}\right).$$

The statement is also valid if BC is parallel to the y-axis or one of the lines $y = \mp x$, and AC is on a steep line, AB is on a gradual line since the rotation with the angle $\pi/2$ or $\pi/4$ is an isometry of the CC-plane [2].

Case III. Assume that AB is on a steep line and AC is on a gradual line. Similar to Case II, one can easily obtain

$$\mathcal{A} = \frac{1}{2}\mathbf{ah} = \frac{\mathbf{a}}{2\sqrt{2}}\left(\mathbf{p}-\mathbf{a}+\frac{\mathbf{c}}{\sqrt{2}}\right).$$

The statement is also valid if BC is parallel to the y-axis or one of the lines $y = \mp x$, and AC is on a steep (gradual) line, AB is on a gradual (steep) line since the rotation with the angle $\pi/2$ or $\pi/4$ is an isometry of the CC-plane [2].

Case IV. Let AB and AC be sides on steep lines. Similarly, one can immediately obtain

$$\mathcal{A} = \frac{1}{2}\mathbf{ah} = \frac{\mathbf{a}}{2}\left(\mathbf{p}-\frac{\mathbf{a}}{\sqrt{2}}\right).$$

Since the rotation with the angle $\pi/2$ or $\pi/4$ is an isometry of CC -plane, the statement valid if BC is parallel to the y-axis, and AB and AC are on gradual lines or if BC is parallel to one of the lines $y = \mp x$, AB is on a steep line and AC is on a gradual line, respectively.

Proposition 3.2. If one side of a triangle ABC, say BC, is parallel to one of the coordinate axes or one of the lines $y = \mp x$ and one of the angles B and C is not an acute angle, say angle B, for the area $\mathcal{A}$ of ABC,

$$\mathcal{A} = \begin{cases} \dfrac{\mathbf{a}}{2(\sqrt{2}-1)}(\mathbf{p}-(\mathbf{a}+\mathbf{a}')), & \text{if } D1 \text{ is satisfied} \\ \dfrac{\mathbf{a}}{2}\left(\mathbf{p}-\mathbf{b}+\dfrac{\mathbf{c}}{\sqrt{2}}\right), & \text{if } D2 \text{ is satisfied} \\ \dfrac{\mathbf{a}}{2}\left(\mathbf{p}-\dfrac{\mathbf{a}}{\sqrt{2}}-(\sqrt{2}-1)\mathbf{a}'\right), & \text{if } D3 \text{ is satisfied,} \end{cases}$$

where A' denotes the foot of the altitude from A and $\mathbf{a}' = d_c(B, A')$,

$D1$: BC is parallel to the x-axis (y-axis), AB, AC are on gradual (steep) lines.

$D2$: BC is parallel to the x-axis (y-axis), AB is on a steep (gradual) line and AC is on a gradual (steep) line or BC is parallel to one of the lines $y = \mp x$, AB and AC are on gradual or steep lines.

$D3$: BC is parallel to the x-axis (y-axis) and AB, AC are on steep (gradual) lines.

Proof. Consider triangle ABC such that BC is parallel to the x-axis and angle B is an obtuse angle. According to positions of sides AB and AC, there are three possible cases.

Case I. Let sides AB and AC be on two gradual lines as shown in Figure 3.

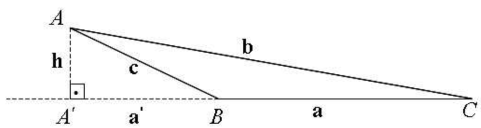

Figure 3.

For the triangles ABA' and ACA', we get

$$\mathbf{c} = \mathbf{a}' + (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{a}' = \mathbf{c} - (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{h} = \frac{\mathbf{c}-\mathbf{a}'}{(\sqrt{2}-1)}$$

and

$$\mathbf{b} = (\mathbf{a}+\mathbf{a}') + (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{b} = \mathbf{a}+\mathbf{c},$$

respectively. From these equalities,

$$\mathbf{p} = \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2} = \mathbf{a}+\mathbf{c}$$

and

$$\mathcal{A} = \frac{1}{2}\mathbf{ah} = \frac{1}{2}\mathbf{a}\left(\frac{\mathbf{c}-\mathbf{a}'}{(\sqrt{2}-1)}\right) = \frac{\mathbf{a}}{2(\sqrt{2}-1)}(\mathbf{p}-(\mathbf{a}+\mathbf{a}')).$$

The statement is also valid if BC is parallel to the y-axis and the sides AB and AC are on steep lines since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

Case II. Assume that side AB is on a steep line and side AC is on a gradual line as shown in Figure 4.

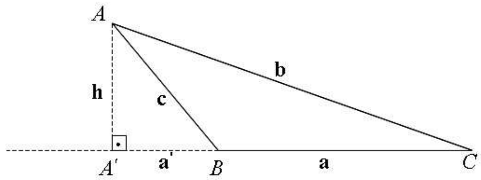

Figure 4.

Using the triangles ABA' and ACA', we obtain

$$\mathbf{c} = \mathbf{h} + (\sqrt{2}-1)\mathbf{a}' \Rightarrow \mathbf{a}' = \frac{\mathbf{c}-\mathbf{h}}{(\sqrt{2}-1)}$$

and

$$\mathbf{b} = (\mathbf{a}+\mathbf{a}') + (\sqrt{2}-1)\mathbf{h} \Rightarrow \mathbf{h} = \frac{\mathbf{a}-\mathbf{b}+(\sqrt{2}-1)\mathbf{c}}{2},$$

respectively. From these equalities,

$$\mathcal{A} = \frac{1}{2}\mathbf{ah} = \frac{1}{2}\mathbf{a}\left(\frac{\mathbf{a}-\mathbf{b}+(\sqrt{2}-1)\mathbf{c}}{2}\right) = \frac{\mathbf{a}}{2}\left(\mathbf{p}-\mathbf{b}+\frac{\mathbf{c}}{\sqrt{2}}\right).$$

The statement is also valid if the BC is parallel to the y-axis, AB is on a gradual line and AC is on a steep line since the rotation with the angle $\pi/2$ is an isometry of the CC-plane. Similarly, the statement is valid if BC is parallel to one of lines $y = \mp x$ and AB, AC are on gradual or steep lines since the rotation with the angle $\pi/4$ is an isometry of the CC-plane.

Case III. Assume that AB and AC are sides on steep lines. Similar to Case I, one can easily obtain

$$\mathcal{A} = \frac{1}{2}\mathbf{ah} = \frac{\mathbf{a}}{2}\left(\mathbf{p}-\frac{\mathbf{a}}{\sqrt{2}}+(\sqrt{2}-1)\mathbf{a}'\right).$$

The argument is also valid if the BC is parallel to the y-axis and AB and AC sides are on gradual lines since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

The following theorem gives the general CC-version of Heron's formula for triangles such that none of their sides are parallel to the coordinate axes.

Theorem 3.3. If ABC is a triangle on the CC-plane, C is the basic vertex, H_1 and H_2 are the feet of the altitudes from the vertices A and B to the base line which is passing through vertex C, respectively and D is the intersection point of the base line and the opposite side of the basic vertex, then area $\mathcal{A}$ of triangle ABC is

$$\mathcal{A} = \begin{cases} \dfrac{\alpha}{2}(2\mathbf{p} - \mathbf{c} - (\sqrt{2}-1)(\alpha_1 + \alpha_2)), & \text{if } E1 \text{ is satisfied} \\ \dfrac{\alpha}{2(\sqrt{2}-1)}(2\mathbf{p} - \mathbf{c} - (\alpha_1 + \alpha_2)), & \text{if } E2 \text{ is satisfied} \\ \dfrac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{a} - \mathbf{c} - (\sqrt{2}-1)(\alpha_1 + (\sqrt{2}+1)^2\alpha_2)), & \text{if } E3 \text{ is satisfied} \\ \dfrac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{b} - \mathbf{c} - (\sqrt{2}-1)((\sqrt{2}+1)^2\alpha_1 + \alpha_2)), & \text{if } E4 \text{ is satisfied,} \end{cases}$$

where $\alpha = d_c(C, D)$, $\alpha_1{=}d_c(C, H_1)$, and $\alpha_2 = d_c(C, H_2)$. Conditions Ei, $i = 1, 2, 3, 4$ are:

$E1$: There is only one base line which is a horizontal (vertical) line, and sides passing through the basic vertex are on steep (gradual) lines or there are two base lines, and sides passing through the basic vertex are on steep lines.

$E2$: There is only one base line which is a horizontal (vertical) line, and sides passing through the basic vertex are on gradual (steep) lines or there are two base lines, and sides passing through the basic vertex are on gradual lines.

$E3$: There is only one base line which is a horizontal (vertical) line, and sides AC and BC are on steep (gradual) and gradual (steep) lines, respectively or there are two base lines, and sides AC and BC are on steep and gradual lines, respectively.

$E4$: There is only one base line which is a horizontal (vertical) line, and sides AC and BC are on gradual (steep) and steep (gradual) lines, respectively or there are two base lines, and sides AC and BC are on gradual and steep lines, respectively.

Proof. At least one of the vertices of the triangle ABC in the plane always has one or two base lines. Let C be the basic vertex of triangle ABC, D be the intersection point of base line and side AB. Let H_1 and H_2 denote the feet of the altitude from vertices of A and B to base line, respectively. Let $\alpha = d_c(C, D)$, $\alpha_1 = d_c(C, H_1)$, $\alpha_2 = d_c(C, H_2)$, and $\mathbf{p}$ be the semiperimeter of triangle ABC. Now, two main cases are possible for base lines.

Case I. Assume that there exist only one base line through the vertex C. In this case according to the slopes of sides of triangle ABC, there are eight possible cases.

Subcase I.1. Let all sides of ABC be on steep lines, and the base line is vertical (see Figure 5). Since sides AC and BC are on steep lines,

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \qquad \text{and} \qquad |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2}-1)}.$$

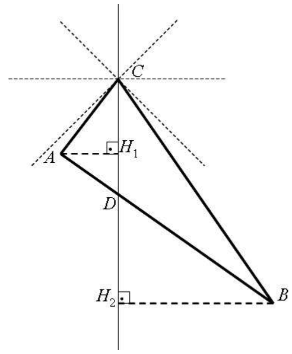

Figure 5.

Therefore,

$$\begin{aligned}\mathcal{A}(ABC) &= \mathcal{A}(ACD) + \mathcal{A}(BCD)\\ &= \frac{|AH_1|\,|CD|}{2} + \frac{|BH_2|\,|CD|}{2}\\ &= \frac{|CD|}{2}(|AH_1| + |BH_2|)\\ &= \frac{|CD|}{2(\sqrt{2}-1)}(|AC| + |BC| - |CH_1| - |CH_2|)\\ &= \frac{|CD|}{2(\sqrt{2}-1)}(2\mathbf{p} - |AB| - |CH_1| - |CH_2|)\\ &= \frac{\alpha}{2(\sqrt{2}-1)}(2\mathbf{p} - \mathbf{c} - (\alpha_1 + \alpha_2)).\end{aligned}$$

The statement is also valid if all sides of ABC are on gradual lines, and the base line is horizontal since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

One can easily obtain the required results for the other subcases similar to subcase I.1.

Subcase I.2. Let $m(AB) < 1$, $m(AC) > 1$, and $m(BC) > 1$ for triangle ABC and the base line is vertical (see Figure 6). Since

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \text{ and } |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2}-1)},$$

we have

$$\mathcal{A}(ABC) = \mathcal{A}(ACD) + \mathcal{A}(BCD) = \frac{\alpha}{2(\sqrt{2}-1)}(2\mathbf{p} - \mathbf{c} - (\alpha_1 + \alpha_2)).$$

Since the rotation with the angle $\pi/2$ is an isometry of the CC-plane, the result is also valid if the base line is horizontal and $m(AB) > 1$, $m(AC) < 1$, and $m(BC) < 1$.

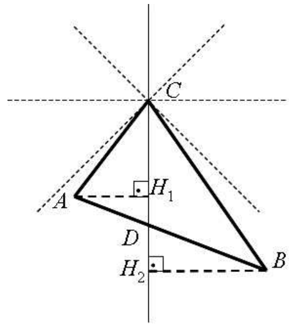

Figure 6.

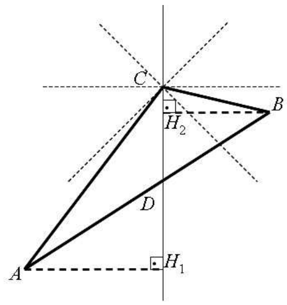

Figure 7.

Subcase I.3. Let $m(AB) > 1$, $m(AC) > 1$, and $m(BC) < 1$ for triangle ABC and the base line is vertical (see Figure 7). As shown in Figure 7

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \text{ and } |BH_2| = |BC| - (\sqrt{2}-1)\,|CH_2|\,.$$

Then

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{b} - \mathbf{c} - (\sqrt{2}-1)((\sqrt{2}+1)^2\alpha_1 + \alpha_2)).$$

Similarly, the statement is valid if the base line is horizontal and $m(AB) < 1$, $m(AC) < 1$, and $m(BC) > 1$ since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

Subcase I.4. Let $m(AB) < 1$, $m(AC) > 1$, and $m(BC) < 1$ for triangle ABC and the base line is vertical (see Figure 8). Since sides AC and BC are on steep and gradual lines, respectively, we obtain

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \text{ and } |BH_2| = |BC| - (\sqrt{2}-1)\,|CH_2|\,.$$

Using these equalities one can obtain

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{b} - \mathbf{c} - (\sqrt{2}-1)((\sqrt{2}+1)^2\alpha_1 + \alpha_2)).$$

The statement is also valid if the base line is horizontal and $m(AB) > 1$, $m(AC) < 1$, and $m(BC) > 1$ since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

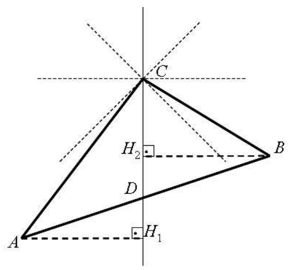

Figure 8.

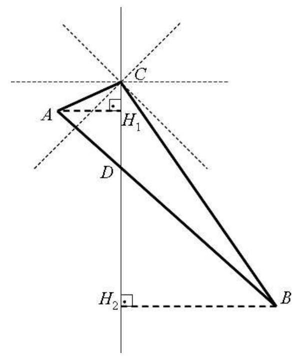

Figure 9.

Subcase I.5. Let $m(AB) > 1$, $m(AC) < 1$, $m(BC) > 1$, and the base line is vertical (see Figure 9). As shown in Figure 9

$$|AH_1| = |AC| - (\sqrt{2} - 1)\,|CH_1| \text{ and } |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2} - 1)}.$$

Then

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{a} - \mathbf{c} - (\sqrt{2} - 1)(\alpha_1 + (\sqrt{2} + 1)^2\alpha_2)).$$

Since the rotation with the angle $\pi/2$ is an isometry of the CC-plane, the result is also valid if the base line is horizontal and $m(AB) < 1$, $m(AC) > 1$, and $m(BC) < 1$.

Subcase I.6. Let $m(AB) < 1$, $m(AC) < 1$, and $m(BC) > 1$ and the base line is vertical (see Figure 10). Since

$$|AH_1| = |AC| - (\sqrt{2} - 1)\,|CH_1| \text{ and } |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2} - 1)},$$

we have

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{a} - \mathbf{c} - (\sqrt{2} - 1)(\alpha_1 + (\sqrt{2} + 1)^2\alpha_2)).$$

The statement is also valid if the base line is horizontal and $m(AB) > 1$, $m(AC) > 1$, and $m(BC) < 1$ since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

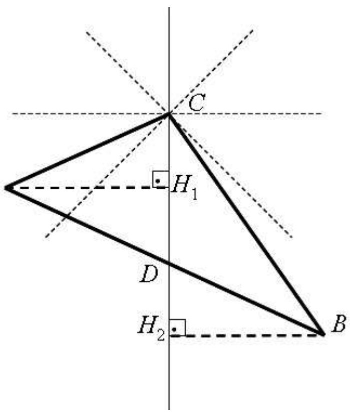

Figure 10.

Subcase I.7. Let $m(AB) > 1$, $m(AC) < 1$, and $m(BC) < 1$ for triangle ABC and the base line is vertical. But there is no triangle satisfying these conditions. Let sides AC and BC be on the lines $y = \frac{1}{c_1}x$ and $y = \frac{-1}{c_2}x$ lines such that $c_1, c_2 \in (1, \infty)$, respectively. Also for $a, b \in \mathbb{R}^+$, $A = \left(-a, \frac{-a}{c_1}\right)$ and $B = \left(b, \frac{-b}{c_2}\right)$. Then the slope of AB must be

$$\frac{\left|\frac{-a}{c_1} + \frac{b}{c_2}\right|}{|a+b|} > 1.$$

This inequality implies that

$$\frac{-a}{c_1} + \frac{b}{c_2} > a + b \text{ or } \frac{a}{c_1} - \frac{b}{c_2} > a + b.$$

From these inequalities, one can get

$$0 > \frac{-a(1+c_1)}{c_1} > \frac{b(c_2-1)}{c_2} > 0 \text{ or } 0 > \frac{a(1-c_1)}{c_1} > \frac{b(1+c_2)}{c_2} > 0.$$

This is a contradiction.

Subcase I.8. Let all the sides of triangle ABC be on gradual lines and the base line be vertical (see Figure 11).

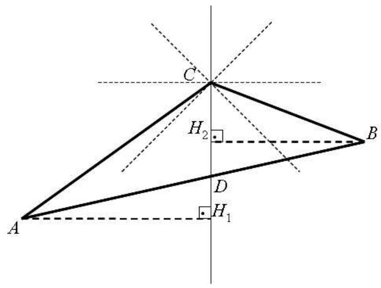

Figure 11.

As shown in Figure 11

$$|AH_1| = |AC| - (\sqrt{2} - 1)\,|CH_1| \;\text{ and }\; |BH_2| = |BC| - (\sqrt{2} - 1)\,|CH_2|\,.$$

Then

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} - \mathbf{c} - (\sqrt{2} - 1)(\alpha_1 + \alpha_2)).$$

Since the rotation with the angle $\pi/2$ is an isometry of the CC-plane, the result is also valid if the base line is horizontal and all sides of ABC are on steep lines.

Case II. Assume that there exist two base lines through the vertex C. Similar to Case I there are eight subcases according to the slopes of the sides of $\triangle ABC$.

Subcase II.1. Let all the sides of $\triangle ABC$ be on step lines (see Figure 12).

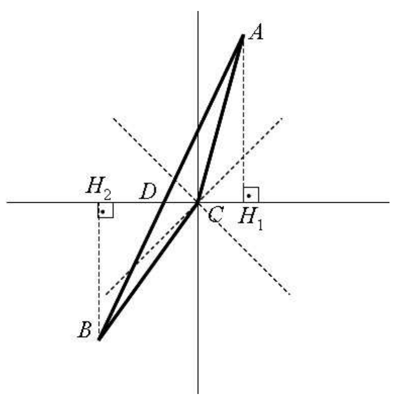

Figure 12.

Since AC and BC are on steep lines, we get

$$|AH_1| = |AC| - (\sqrt{2}-1)\,|CH_1| \text{ and } |BH_2| = |BC| - (\sqrt{2}-1)\,|CH_2|\,.$$

Therefore, one can get

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} - \mathbf{c} - (\sqrt{2}-1)(\alpha_1 + \alpha_2)).$$

The result is also valid if all sides of ABC are on gradual lines since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

Subcase II.2. Let $m(AB) < 1$, $m(AC) > 1$, and $m(BC) > 1$. But there is no triangle satisfying these conditions. Let AC and BC be on $y = c_2x$ and $y = c_1x$ lines such that $c_1, c_2 \in (1,\infty)$, respectively. Also $A = (-a, -c_1a)$ and $B = (b, c_2b)$ for $a, b \in \mathbb{R}^+$. Then the slope of AB must be

$$\frac{|c_1a + c_2b|}{|a+b|} < 1.$$

This implies that $|c_1a + c_2b| < |a + b|$. This is a contradiction.

Subcase II.3. Let $m(AB) > 1$, $m(AC) > 1$, and $m(BC) < 1$ (see Figure 13). Since

$$|AH_1| = |AC| - (\sqrt{2}-1)\,|CH_1| \text{ and } |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2}-1)},$$

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{a} - \mathbf{c} - (\sqrt{2}-1)(\alpha_1 + (\sqrt{2}+1)^2\alpha_2)).$$

Since the rotation with the angle $\pi/2$ is an isometry of the CC-plane, the result is also valid if $m(AB) < 1$, $m(AC) < 1$, and $m(BC) > 1$.

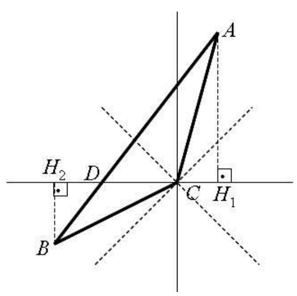

Figure 13.

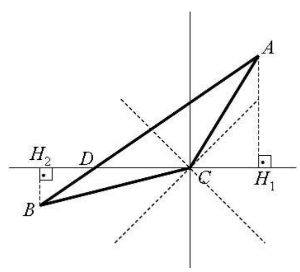

Figure 14.

Subcase II.4. Let $m(AB) < 1$, $m(AC) > 1$, and $m(BC) < 1$ for triangle ABC (see Figure 14). As shown in Figure 14

$$|AH_1| = |AC| - (\sqrt{2}-1)\,|CH_1| \text{ and } |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2}-1)}.$$

Then

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{a} - \mathbf{c} - (\sqrt{2}-1)(\alpha_1 + (\sqrt{2}+1)^2\alpha_2)).$$

The statement is also valid if $m(AB) > 1$, $m(AC) < 1$, and $m(BC) > 1$ since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

Subcase II.5. Let $m(AB) > 1$, $m(AC) < 1$, and $m(BC) > 1$ for triangle ABC (see Figure 15). Since sides AC and BC are on gradual and steep lines, respectively, we have

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \text{ and } |BH_2| = |BC| - (\sqrt{2}-1)\,|CH_2|\,.$$

Using these equalities one can get

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{b} - \mathbf{c} - (\sqrt{2}-1)((\sqrt{2}+1)^2\alpha_1 + \alpha_2)).$$

Similarly, the statement is valid if $m(AB) < 1$, $m(AC) > 1$, and $m(BC) < 1$ since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

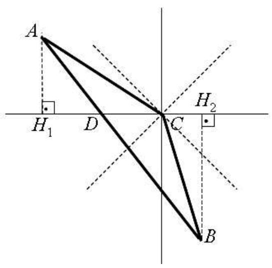

Figure 15.

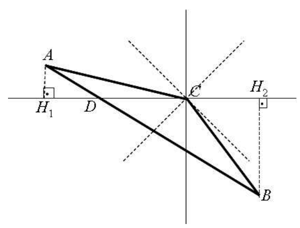

Figure 16.

Subcase II.6. Let $m(AB) < 1$, $m(AC) < 1$, and $m(BC) > 1$ (see Figure 16). Since

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \text{ and } |BH_2| = |BC| - (\sqrt{2}-1)\,|CH_2|\,,$$

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} + \sqrt{2}\mathbf{b} - \mathbf{c} - (\sqrt{2}-1)((\sqrt{2}+1)^2\alpha_1 + \alpha_2)).$$

The result is also valid if $m(AB) > 1$, $m(AC) > 1$, and $m(BC) < 1$ since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

Subcase II.7. Let $m(AB) > 1$, $m(AC) < 1$, and $m(BC) < 1$ for ABC. But there is no triangle satisfying these conditions. Let AC and BC be on $y = \frac{-x}{c_1}$ and $y = \frac{-x}{c_2}$ lines such that $c_1, c_2 \in (1, \infty)$, respectively. Also,

$$A = \left(-a, \frac{a}{c_1}\right) \text{ and } B = \left(b, \frac{-b}{c_2}\right)$$

for $a, b \in \mathbb{R}^+$. Then the slope of AB must be

$$\frac{\left|\frac{a}{c_1} + \frac{b}{c_2}\right|}{|a+b|} > 1.$$

This implies that $c_2 a + c_1 b > c_1 \cdot c_2\,(a+b)$. This is a contradiction.

Subcase II.8. Let all sides of ABC be on gradual lines (see Figure 17).

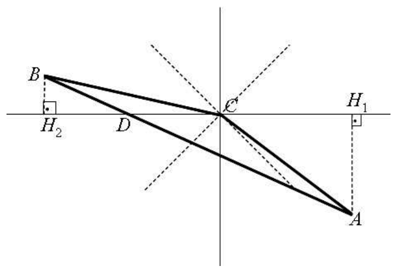

Figure 17.

Since sides AC and BC are on gradual lines, we obtain

$$|AH_1| = \frac{|AC| - |CH_1|}{(\sqrt{2}-1)} \text{ and } |BH_2| = \frac{|BC| - |CH_2|}{(\sqrt{2}-1)}.$$

Therefore, one can obtain

$$\mathcal{A}(ABC) = \frac{\alpha}{2}(2\mathbf{p} - \mathbf{c} - (\sqrt{2}-1)(\alpha_1 + \alpha_2)).$$

The statement is also valid if all sides of triangle ABC are on steep lines since the rotation with the angle $\pi/2$ is an isometry of the CC-plane.

Considering the results of all the cases and subcases, we establish the CC-version of the Heron's formula.

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