A CHARACTERIZATION OF HIGHER ORDER WIELANDT SUBGROUPS AND SOME APPLICATIONS

A. Y. M. CHIN AND ANGELINE P. L. LEE

ABSTRACT. In this paper we obtain a characterization of the ith Wielandt subgroup of a finite group. We then use this to find relations between the Wielandt length of a finite group and that of some of its factor groups. This in turn is used to obtain an upper bound for the Wielandt lengths of finite p-groups.

1. INTRODUCTION

The *Wielandt subgroup* of a group G, denoted by $\omega(G)$, was originally defined by Helmut Wielandt in a paper published in 1958 [2]. It is defined as the intersection of the normalizers of all the subnormal subgroups of G. Recall that a subgroup H of G is called *subnormal* if there exists a normal series from H to G. The notation $H \lhd\lhd G$ is used to denote that H is a subnormal subgroup of G.

In the same paper, Wielandt showed that if G is a group satisfying the minimal condition on subnormal subgroups, then $\omega(G)$ contains every minimal normal subgroup of G, and hence contains the socle of G; thus implying that $\omega(G) \neq \{1\}$. In addition to this, Wielandt [2] showed that $\omega(G)$ is a characteristic subgroup of G when G is finite. This additional property enables the construction of an ascending normal series of G when G is finite: Let $\omega_0(G) = \{1\}$, $\omega_1(G) = \omega(G)$, and for any integer $i \geq 2$, let $\omega_i(G)$ be the subgroup of G such that $\omega_i(G)/\omega_{i-1}(G) = \omega\left(G/\omega_{i-1}(G)\right)$. Since G is finite, there exists a smallest integer n such that $\omega_n(G) = G$.

In a paper published in 1970, Camina [1] introduced the notion of the Wielandt length of a group. For a finite group G, its *Wielandt length* $wl\,(G)$ is just the smallest positive integer n such that $\omega_n(G) = G$.

In this paper we obtain a characterization of the ith Wielandt subgroup of a finite group. This characterization, although not unexpected, does not seem to be available in the literature. We demonstrate its usefulness by using it to find relations between the Wielandt length of a finite group and that of some of its factor groups. As an application, we then obtain an upper bound for the Wielandt lengths of finite p-groups.

2. A Characterization

Lemma 2.1. *Let G be a group, H a normal subgroup of G, and K a subgroup of G with $H \subseteq K \subseteq G$. Then*

(i) $N_{G/H}(K/H) = N_G(K)\big/H$;

(ii) $K \lhd\lhd G$ *if and only if* $K/H \lhd\lhd G/H$.

Proof. (i) Let $xH \in N_{G/H}(K/H)$. Then for any $k \in K$, $x^{-1}kx = k'h$ for some $k' \in K$ and $h \in H$. But since $H \subseteq K$, we have $x^{-1}kx \in K$. Since k is arbitrary, it follows that $x \in N_G(K)$ and hence, $xH \in N_G(K)/H$. For the reverse inclusion, if $xH \in N_G(K)/H$, then $x \in N_G(K)$ and hence, $(xH)^{-1}(kH)(xH) = x^{-1}kxH \in K/H$ for any $kH \in K/H$. Thus, $xH \in N_{G/H}(K/H)$, as required.

(ii) Suppose that $K \lhd\lhd G$. Then there exists a normal series from K to G, say

$$K = K_0 \lhd K_1 \lhd K_2 \lhd \cdots \lhd K_n \lhd G. \tag{1}$$

Since H is a normal subgroup of G contained in K, so H is also normal in K_i ($i = 0, 1, \ldots, n$) and we thus have by (1) the normal series

$$K/H = K_0/H \lhd K_1/H \lhd K_2/H \lhd \cdots \lhd K_n/H \lhd G/H \tag{2}$$

from K/H to G/H. This shows that $K/H \lhd\lhd G/H$. Conversely, any normal series of the form (2) will give rise to a normal series of the form (1). Thus, $K \lhd\lhd G$ if $K/H \lhd\lhd G/H$. □

Using Lemma 2.1 we obtain a characterization of the ith Wielandt subgroup of a finite group as follows.

Proposition 2.2. *Let G be a finite group. Then for any integer $i \geq 0$,*

$$\omega_{i+1}(G) = \cap\left\{N_G(K) \mid \omega_i(G) \leq K \lhd\lhd G\right\}.$$

Proof. The case $i = 0$ follows readily from the definition of $\omega(G)$. Now suppose that i is an integer ≥ 1. Note that

$$\begin{aligned}
&\omega_{i+1}(G)/\omega_i(G)\\
= \;& \omega\left(G/\omega_i(G)\right) \quad \text{(by definition)}\\
= \;& \cap\left\{N_{G/\omega_i(G)}\left(K/\omega_i(G)\right) \mid K/\omega_i(G) \lhd\lhd G/\omega_i(G)\right\} \quad \text{(by definition)}\\
= \;& \cap\left\{N_G(K)/\omega_i(G) \mid K/\omega_i(G) \lhd\lhd G/\omega_i(G)\right\} \quad \text{(by Lemma 2.1(i))}\\
= \;& \cap\left\{N_G(K)/\omega_i(G) \mid \omega_i(G) \leq K \lhd\lhd G\right\} \quad \text{(by Lemma 2.1(ii))}\\
= \;& \cap\left\{N_G(K) \mid \omega_i(G) \leq K \lhd\lhd G\right\}\Big/\omega_i(G).
\end{aligned}$$

Hence, $\omega_{i+1}(G) = \cap \{N_G(K) \mid \omega_i(G) \leq K \lhd\lhd G\}$ also holds for every integer $i \geq 1$. □

3. Relations Between Wielandt Length of a Group and its Factor Groups

In this section, we show how the Wielandt lengths of finite groups are related to the Wielandt lengths of certain factor groups and establish an upper bound for the Wielandt lengths of finite p-groups.

Proposition 3.1. *Let G be a finite group and let n be a fixed positive integer. Then $\omega_{n-m}\Big(G/\omega_m(G)\Big) = \omega_n(G)/\omega_m(G)$ for all integers m with $0 \leq m \leq n$.*

Proof. If $m = n$, the result is obvious. Suppose that the result is true for $k = n - m$ (≥ 0), that is,

$$\omega_k\left(G/\omega_m(G)\right) = \omega_{m+k}(G)/\omega_m(G). \tag{3}$$

Then

$$\begin{aligned}
&\omega_{k+1}\left(G/\omega_m(G)\right)\\
&= \cap \left\{N_{G/\omega_m(G)}\left(K/\omega_m(G)\right) \mid \omega_k\left(G/\omega_m(G)\right) \leq K/\omega_m(G) \lhd\lhd G/\omega_m(G)\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Proposition 2.2)}\\
&= \cap \{N_G(K)/\omega_m(G) \mid \omega_{m+k}(G)/\omega_m(G) \leq K/\omega_m(G) \lhd\lhd G/\omega_m(G)\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 2.1(i) and (3))}\\
&= \cap \{N_G(K) \mid \omega_{m+k}(G) \leq K \lhd\lhd G\}/\omega_m(G)\\
&= \omega_{m+k+1}(G)/\omega_m(G) \qquad\qquad\qquad\qquad \text{(by Proposition 2.2).}
\end{aligned}$$

The assertion thus follows by induction. □

Theorem 3.2. *Let G be a finite group and let $m \geq 0$, $n > 0$ be integers. If the Wielandt length of $G/\omega_m(G)$ is n, then the Wielandt length of G is $m + n$.*

Proof. The result is clearly true if $m = 0$. We thus assume that $m > 0$. Suppose that the Wielandt length of $G/\omega_m(G)$ is n. Then by Proposition 3.1, we have

$$\omega_{n+m}(G)\big/\omega_m(G) = \omega_n\big(G/\omega_m(G)\big) = G\big/\omega_m(G)$$

which implies that $\omega_{n+m}(G) = G$. Hence, the Wielandt length of G is at most $n + m$. We claim that the Wielandt length of G is exactly $n + m$. Indeed, if $\omega_{n+m-1}(G) = G$, then by Proposition 3.1,

$$G\big/\omega_m(G) = \omega_{n+m-1}(G)\big/\omega_m(G) = \omega_{n-1}\big(G/\omega_m(G)\big).$$

But it follows from this that the Wielandt length of $G/\omega_m(G)$ is at most $n-1$ which is a contradiction. Thus, $\omega_{n+m-1}(G) \neq G$ and this immediately implies that G must have Wielandt length $n+m$. □

It is known that if G is a p-group of order p^n $(n \geq 2)$, then the nilpotency class of G is at most $n-1$. A similar relationship exists between the order of G and its Wielandt length, as shown below.

Proposition 3.3. *Let G be a group of order p^n for some integer $n \geq 2$. Then G has Wielandt length less than or equal to $n-1$.*

Proof. If $n = 2$, then G is abelian and hence, has Wielandt length one which agrees with the assertion.

Now assume that any group of order $p^k(k \geq 2)$ has Wielandt length at most $k-1$. Consider the case $|G| = p^{k+1}$. Since $\omega(G)$ is nontrivial, $|G/\omega(G)| \leq p^k$. Hence, by the induction hypothesis, $G/\omega(G)$ has Wielandt length at most $k-1$. By Theorem 3.2, G has Wielandt length at most $(k-1)+1 = k$ which completes the proof. □

References

[1] A. R. Camina, *The Wielandt Length of Finite Groups*, J. Algebra, **15** (1970), 142–148.

[2] H. Wielandt, *Über den Normalisator der Subnormalen Untergruppen*, Math. Z., **69** (1958), 463–465.

MSC2000: 20D25, 20D35

Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia

E-mail address: acym@um.edu.my

Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia