# A GENERALIZED MARTINGALE BETTING STRATEGY

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Abstract. A generalized martingale betting strategy is analyzed for which bets are increased by a factor of $m \geq 1$ after each loss, but return to the initial bet amount after each win. The average amount bet and the average final fortune are derived for sequences of $n$ bets, for the number of bets $T$ that results in the first win, and for $\min(T,\ n)$.

## 1. Introduction

In casino gambling, the bold strategy is the process of always betting your entire stake or just enough to reach your target goal, whichever is least. It is argued in [2] and [5] that this method is the optimal strategy in terms of minimizing the probability of ruin. The classical martingale strategy is to double your bet after each loss until you either win or have insufficient funds to continue doubling the bet. If your goal is simply to "get ahead" and you receive a 1:1 payoff, then doubling an initial bet of $1 is equivalent to the bold strategy. But with an 8:1 payoff, there would be no need to increase the bet until after the eighth loss. By doubling the bets after each loss, very large bets may eventually need to be made which often are not allowed due to table limits. The bold strategy that accounts for house limits is discussed in [3]. Other examples of bold play with regard to ruin are discussed in [1] and [4].

But is there really any need to double the bets? Why not choose a smaller factor $m$ so that the bet sizes do not increase so quickly and do not run afoul of table limits? In this paper, we shall analyze a more general scenario for which you multiply your last bet by a factor of $m \geq 1$ after each loss, but use the initial bet amount after each win. Some interesting questions arise using this strategy. Namely, what is the average amount bet on the $k$th wager? What is your average fortune after a sequence of bets? How do we choose $m$ so that we are guaranteed to come out ahead upon any win? Most importantly, to have a high probability of coming out ahead, how many bets would be needed and what initial stake would be needed?

If we are wise, then we will quit whenever we get ahead, which should occur after the first win with the right choice of $m$ using this general martingale strategy. So suppose we stop upon the first win. What would be the average amount bet on this last wager, and what would be the average fortune after this last bet that resulted in the first win? Finally, suppose that we stop upon the first win or after a maximum of $n$ bets, whichever comes first. Now what are the average amount of the last bet and the average fortune upon stopping?

In this article we shall answer these questions using general probabilistic concepts and geometric series. We then shall compare the results of this general martingale strategy with the results obtained from some other basic strategies.

## 2. General Betting Notation

Throughout, we shall let $X_n$ denote the bettor's fortune after $n$ bets under our general martingale strategy, where $X_0$ is a random initial stake. There is probability $p$ of winning each bet and probability $q = 1-p$ of losing. A typical sequence of wins and losses will be denoted by $\omega = (\omega_1, \ldots, \omega_n)$, where $\omega_i = +1$ to designate a win or $-1$ to designate a loss. The initial bet is $b_1 = \$b$, which is paid only if a loss occurs, and the initial payoff for winning is $a_1 = \$a$. After the initial wager, the successive amounts bet are dependent upon the previous outcome. If there was a win on the $(k-1)$st bet, then $b_k$ returns to $b$. But if there was a loss on the $(k-1)$st bet, then $b_k = mb_{k-1}$, for a fixed multiplicative factor $m \geq 1$. Thus, for $k \geq 2$,

$$b_k(\omega) = b1_{\{\omega_{k-1}=1\}} + mb_{k-1}(\omega)1_{\{\omega_{k-1}=-1\}}. \tag{1}$$

We note that, although the successive amounts bet are dependent on the previous outcome, the last actual outcome $\omega_{k-1}$ of a win or loss is *independent* of the amount $b_{k-1}$ that had been bet.

If we were to bet the *same* amount $b$ each time (i.e., use $m = 1$), then the average change of fortune after each bet would be $ap - bq$. The average fortune in this case after $n$ such bets is then

$$E[X_n] = E[X_0] + n(ap - bq). \tag{2}$$

Because casinos operate to make a profit in the long run, they desire to have $E[X_n] < E[X_0]$, which occurs if and only if $ap - bq < 0$. So the casino must choose a non-advantageous payoff $a$ that satisfies $a < (q/p)b$. Under our general martingale strategy, we shall assume this condition as well as a fixed payoff ratio so that the payoff $a_k$ on the $k$th bet satisfies $a_k/b_k = a/b$.

Thus, $a_k = (a/b)b_k$ which from Equation (1) gives, for $k \geq 2$,

$$a_k(\omega) = a1_{\{\omega_{k-1}=1\}} + m\left(\frac{a}{b}\right) b_{k-1}(\omega)1_{\{\omega_{k-1}=-1\}}. \tag{3}$$

## 3. The Average Amounts Bet

Because $\omega_{k-1}$ is independent of $b_{k-1}$ and the average of an independent product is the product of the averages, the average amount bet on the $k$th wager, for $k \geq 2$, can be written as

$$E[b_k] = bE[1_{\{\omega_{k-1}=1\}}] + mE[b_{k-1}]E[1_{\{\omega_{k-1}=-1\}}] = bp + mqE[b_{k-1}].$$

Thus, $E[b_1] = b$ and $E[b_2] = bp + mqb$, while

$$E[b_3] = bp + mqE[b_2] = bp(1+mq) + (mq)^2\, b$$

and

$$E[b_4] = bp + mqE[b_3] = bp(1+mq+(mq)^2) + (mq)^3\, b\,.$$

By an inductive argument, we have for $k \geq 2$,

$$E[b_k] = bp\sum_{i\,=\,0}^{k-2}(mq)^i + (mq)^{k-1}\, b\,. \tag{4}$$

If $mq = 1$, then Equation (4) simplifies to $(k-1)bp + b$. If $mq \neq 1$, then we can simplify (4) using the geometric series formula $\sum_{i\,=\,0}^{n} x^i = (1-x^{n+1})/(1-x)$ for $x \neq 1$. We thereby obtain our first result.

**Theorem 1.** *Under the conditions of the general martingale strategy when repeated rounds are played, the average amount bet on the $k$th wager is*

$$E[b_k] = \begin{cases} bp(k-1) + b & \text{if } mq = 1 \\[2ex] \dfrac{bp(1-(mq)^{k-1})}{1-mq} + (mq)^{k-1}\, b & \text{if } mq \neq 1. \end{cases}$$

Because $q = 1-p$, $E[b_k]$ simplifies to $b$ when $m = 1$, and $E[b_k]$ is clearly $b$ when $k = 1$.

## 4. Stopping the Betting Process

Suppose that betting stops upon the first win and $T(\omega)$ denotes the number of bets needed for string $\omega$, where $T((-1,-1,-1,\ldots)) = +\infty$. Then $T$ is a geometric random variable so that $P(T = k) = q^{k-1}p$ and $P(T \leq k) = 1 - q^k$ for $k \geq 1$. Then $b_T$ is the amount of the bet that gave the first win. In the case of all losses, we let $b_\infty = b$ if $m = 1$ and let $b_\infty = +\infty$ if $m > 1$. We now shall derive $E[b_T]$, the average of all bets that give the first win. For stopping upon the first win or after a total of $n$ bets, whichever comes first, we shall find $E[b_{T\wedge n}]$, the average of the last amounts bet where $T(\omega) \wedge n = \min(T(\omega),\ n)$.

If $q = 1$, then there is only the string of all losses; hence, $b_T \equiv b_\infty$ and $E[b_T]$ is either $b$ or $+\infty$, depending on $m$. For $q < 1$, the string of all losses has probability 0, so $b_T$ can be written almost surely as

$$b_T = \sum_{i=1}^{\infty} m^{i-1}\, b\, 1_{\{T=i\}}.$$

If we only allow a maximum of $n$ bets, then for all $q$ and all $\omega$,

$$b_{T\wedge n} = \sum_{i=1}^{n-1} m^{i-1}\, b\, 1_{\{T=i\}} + m^{n-1}\, b\, 1_{\{T\geq n\}}.$$

Taking the expected value of $b_T$ (for $q < 1$), we obtain

$$E[b_T] = b\sum_{i=1}^{\infty} m^{i-1} P(T=i) = b\sum_{i=1}^{\infty} m^{i-1} q^{i-1} p = bp\sum_{i=0}^{\infty} (mq)^i,$$

and taking the expected value of $b_{T\wedge n}$ gives

$$\begin{aligned} E[b_{T\wedge n}] &= b\sum_{i=1}^{n-1} m^{i-1} P(T=i) + m^{n-1}\, b\, P(T\geq n) \\ &= bp\sum_{i=0}^{n-2} (mq)^i + m^{n-1}\, b\, q^{n-1}. \end{aligned}$$

Simplifying the geometric series, we obtain the following closed forms for the desired averages.

**Theorem 2.** *Let $T$ be the number of bets needed for the first win. Under the conditions of the general martingale strategy, the average amounts bet on the $T$th bet and on the $(T\wedge n)$th bet are*

$$E[b_T] = \begin{cases} b & \text{if } m = 1 \\ +\infty & \text{if } m > 1 \text{ and } mq \geq 1 \\ \dfrac{bp}{1-mq} & \text{if } m > 1 \text{ and } mq < 1 \end{cases}$$

*and*

$$E[b_{T\wedge n}] = \begin{cases} bp(n-1) + b & \text{if } mq = 1 \\ \dfrac{bp(1-(mq)^{n-1})}{1-mq} + b(mq)^{n-1} & \text{if } mq \neq 1. \end{cases}$$

It can easily be verified that $\lim_{n\to\infty} E[b_{T\wedge n}] = E[b_T]$ by direct evaluation of the limit. However, the result also follows from the Monotone Convergence Theorem. Indeed, $T(\omega)\wedge n = n$ for $n < T(\omega)$, but $T(\omega)\wedge n = T(\omega)$ for all $n \geq T(\omega)$; thus, $\lim_{n\to\infty} b_{T(\omega)\wedge n} = b_{T(\omega)}$ for all $\omega$. Moreover, because we assume $m \geq 1$, the sequential amounts bet are increasing: $b_1 \leq b_2 \leq \cdots \leq b_T$. That is, $b_{T\wedge n}$ increases almost surely to $b_T$; thus, $\lim_{n\to\infty} E[b_{T\wedge n}] = E[b_T]$.

Comparing the results of Theorems 1 and 2, we also can see that $E[b_{T\wedge n}] = E[b_n]$ for all $n$, even though $b_{T\wedge n}$ and $b_n$ are different functions. In fact, $b_{T\wedge n}$ and $b_n$ are identically distributed as we next prove.

**Theorem 3.** *For all $n \geq 1$, $b_{T\wedge n}$ and $b_n$ have the same distribution.*

*Proof.* If $1 \leq T(\omega) \leq n-1$, then the first win occurred before $n$ bets were made. Letting $T(\omega) = k$, then there were $k-1$ initial losses followed by the win on the $k$th bet, which occurs with probability $q^{k-1}p$. Because of the $k-1$ straight losses, the amount of this last bet was $m^{k-1}b$. If there were $n-1$ losses in a row, which occurs with probability $q^{n-1}$, then $T(\omega)\wedge n = n$ and the amount of the $n$th and last bet is $m^{n-1}b$. Thus, there are precisely $n$ values in the range of $b_{T\wedge n}$ which is the set $\{b, \ldots, m^{n-1}b\}$ and

$$P(b_{T\wedge n} = m^j b) = \begin{cases} q^j p & \text{for } 0 \leq j \leq n-2 \\ \\ q^{n-1} & \text{for } j = n-1. \end{cases}$$

For a complete sequence of $n$ bets, the amount placed on the $n$th bet depends on how many losses in a row preceded that bet. If there were $n-1$ losses in a row, which occurs with probability $q^{n-1}$, then $b_n = m^{n-1}b$. If there were fewer losses in a row just before the $n$th bet, say $j$ losses for $0 \leq j \leq n-2$, then a win must have immediately preceded the string of losses. This event occurs with probability $pq^j$ and occurs if and only if $b_n = m^j b$. Thus, $b_n$ has the same range as $b_{T\wedge n}$ and $P(b_n = m^j b) = P(b_{T\wedge n} = m^j b)$ for $0 \leq j \leq n-1$. □

In other words, you may get your first win before making $n$ bets and stop betting at that point. Or you might make a full sequence of $n$ bets where each win throughout causes the next bet to go back to \$$b$. Either way, the possibilities for your last bet, $b_{T\wedge n}$ or $b_n$, are precisely the same and $E[b_{T\wedge n}] = E[b_n]$. However, we shall see that the average fortunes after these last bets, $E[X_{T\wedge n}]$ and $E[X_n]$, are *not* the same.

## 5. The Average Fortune After $n$ Bets

With our general martingale strategy, the fortune after the $n$th bet has either increased by an amount of $a_n = (a/b)b_n$ from the fortune after the previous bet, or has decreased by an amount of $b_n$. So for $n \geq 1$, $X_n$ can be written as

$$X_n(\omega) = X_{n-1}(\omega) + \left(\frac{a}{b}\right) b_n(\omega) 1_{\{\omega_n = 1\}} - b_n(\omega) 1_{\{\omega_n = -1\}}.$$

Because the amount bet $b_n$ is independent of whether or not this bet is won $\omega_n$, we have

$$\begin{aligned} E[X_n] &= E[X_{n-1}] + \left(\frac{a}{b}\right) E[b_n] E[1_{\{\omega_n=1\}}] - E[b_n] E[1_{\{\omega_n=-1\}}] \\ &= E[X_{n-1}] + E[b_n]\left(\frac{a}{b}p - q\right). \end{aligned} \tag{5}$$

Applying recursion, we obtain

$$\begin{aligned} E[X_n] &= E[X_{n-2}] + (E[b_n] + E[b_{n-1}])\left(\frac{a}{b}p - q\right) \\ &= E[X_{n-3}] + (E[b_n] + E[b_{n-1}] + E[b_{n-2}])\left(\frac{a}{b}p - q\right) \\ &= \cdots \\ &= E[X_0] + \left(\frac{a}{b}p - q\right)\sum_{k=1}^{n} E[b_k]. \end{aligned} \tag{6}$$

If $m = 1$, then $E[b_k] = b$, which gives $E[X_n] = E[X_0] + n(ap - bq)$. Otherwise, we use the result of Theorem 1 to obtain

$$E[X_n] = \begin{cases} E[X_0] + \left(\frac{a}{b}p - q\right)\sum\limits_{k=1}^{n}(bp(k-1) + b) & \text{if } mq = 1 \\ E[X_0] + \left(\frac{a}{b}p - q\right)\sum\limits_{k=1}^{n}\left(\frac{bp(1-(mq)^{k-1})}{1-mq} + (mq)^{k-1}\, b\right) & \text{if } mq \neq 1. \end{cases}$$

Simplifying this result gives us our next desired average.

**Theorem 4.** *Under the conditions of the general martingale strategy when repeated rounds are played, the average fortune after a sequence of $n$ bets is*

$$E[X_n] = \begin{cases} E[X_0] + n(ap - bq)\left(\dfrac{(n-1)p}{2} + 1\right) & \textit{if } mq = 1 \\ E[X_0] + \left(\dfrac{ap - bq}{1 - mq}\right)\left(np + \dfrac{q(1-m)(1-(mq)^n)}{1 - mq}\right) & \textit{if } mq \neq 1. \end{cases}$$

We note that for $m = 1$, the expressions simplify to $E[X_0] + n(ap - bq)$ in both resulting cases of $q = 1$ and $q \neq 1$. Moreover, because of our assumption that $ap - bq < 0$, we see from Equations (5) and (6) that $E[X_n] < E[X_{n-1}]$ and $E[X_n] < E[X_0]$ for all $n \geq 1$.

## 6. The Average Fortune Upon Winning

We again let $T$ denote the number of bets needed to attain the first win. Then $X_T$ denotes the final fortune when stopping after the first win no matter when it occurs. But if we stop upon winning or making a total of $n$ bets, then $X_{T\wedge n}$ denotes the final fortune. We first shall derive $E[X_T]$. This average depends on several conditions regarding the terms $m$ and $mq$. But in all cases, we will assume that $q < 1$ so that $T < \infty$ with probability 1.

Suppose we win for the first time on the $i$th bet. Then the first $i-1$ wagers all resulted in losses which means that $b_j = m^{j-1}b$ for $1 \le j \le i$. The payoff for winning when $T = i$ is $a_i = (a/b)b_i = m^{i-1}a$. We now let $D_i$ denote the total deficit accrued with the initial $i-1$ losses. Then

$$D_i = \sum_{j=1}^{i-1} m^{j-1}b = \begin{cases} (i-1)b & \text{if } m = 1 \\ \\ b \times \dfrac{m^{i-1}-1}{m-1} & \text{if } m > 1. \end{cases} \tag{7}$$

Hence, the fortune after the first win can be written (almost surely) as

$$\begin{aligned} X_T &= X_0 + \sum_{i=1}^{\infty} \left(m^{i-1}a - D_i\right) 1_{\{T=i\}} \qquad (8) \\ &= \begin{cases} X_0 + a - b \sum_{i=1}^{\infty} (i-1)1_{\{T=i\}} & \text{if } m = 1 \\ \\ X_0 + \dfrac{b}{m-1} + \sum_{i=1}^{\infty} \left(\left(a - \dfrac{b}{m-1}\right) m^{i-1}\right) 1_{\{T=i\}} & \text{if } m > 1. \end{cases} \end{aligned}$$

Taking the expected value when $m = 1$, we obtain

$$\begin{aligned} E[X_T] &= E[X_0] + a - b\sum_{i=1}^{\infty} i\,P(T=i) + b\sum_{i=1}^{\infty} P(T=i) \\ &= E[X_0] + a - bE[T] + b \\ &= E[X_0] + a - b\left(\frac{1}{p} - 1\right) \\ &= E[X_0] + \frac{ap - bq}{p}\,. \end{aligned} \tag{9}$$

Several cases arise if $m > 1$. The first case is when $a - b/(m-1) = 0$, which is equivalent to $m = 1 + b/a$. This situation occurs for instance with 1:1 payoffs and doubling the bets so that $a = b$ and $m = 2$. In this case, the infinite series term in $X_T$ in (8) is identically 0, and $b/(m-1) = a$; hence, $X_T \equiv X_0 + a$ and $E[X_T] = E[X_0] + a$.

Secondly, because $a < (q/p)b$, then $m \geq 1+b/a$ implies $mq \geq q+(b/a)q > q+p = 1$. So in the case of $m > 1$ but $mq < 1$, we cannot have $m = 1+b/a$. In this case we have

$$\begin{aligned}
E[X_T] &= E[X_0] + \frac{b}{m-1} + \left(a - \frac{b}{m-1}\right) \sum_{i=1}^{\infty} m^{i-1} P(T=i) \\
&= E[X_0] + \frac{b}{m-1} + \left(a - \frac{b}{m-1}\right) p \sum_{i=0}^{\infty} (mq)^i \\
&= E[X_0] + \frac{b}{m-1} + \left(a - \frac{b}{m-1}\right)\left(\frac{p}{1-mq}\right) \\
&= E[X_0] + \frac{ap}{1-mq} + \frac{b(1-mq)-bp}{(m-1)(1-mq)} \\
&= E[X_0] + \frac{ap}{1-mq} + \frac{b(1-p)-bmq}{(m-1)(1-mq)} \\
&= E[X_0] + \frac{ap}{1-mq} + \frac{bq(1-m)}{(m-1)(1-mq)} \\
&= E[X_0] + \frac{ap-bq}{1-mq}, \qquad (10)
\end{aligned}$$

which gives the same result as in Equation (9) if we were to let $m = 1$.

Thirdly, if $m > 1 + b/a$ (which implies $mq > 1$), then $E[X_T]$ is as in the beginning of Equation (10), but $\sum_{i=0}^{\infty}(mq)^i$ diverges to $+\infty$. Thus, $E[X_T] = +\infty$ because $a - b/(m-1) > 0$. Finally, if $1 < m < 1 + b/a$ and $mq \geq 1$, then $\sum_{i=0}^{\infty}(mq)^i = +\infty$, but $E[X_T] = -\infty$ because $a-b/(m-1) < 0$.

Collecting the results in (9) and (10) and considering all cases, we obtain our formulization for $E[X_T]$.

**Theorem 5.** *Let $T$ be the number of bets needed for the first win. Under the conditions of the general martingale strategy with $q < 1$ and $a < (q/p)b$, the average fortune after the $T$th bet is*

$$E[X_T] = \begin{cases} E[X_0] + \dfrac{ap-bq}{1-mq} & \text{if } m = 1, \text{ or if } m > 1 \text{ with } mq < 1 \\ E[X_0] + a & \text{if } m = 1 + b/a \\ +\infty & \text{if } m > 1 + b/a \text{ (so that } mq > 1\text{)} \\ -\infty & \text{if } 1 < m < 1 + b/a \text{ and } mq \geq 1. \end{cases}$$

We thereby obtain two cases for which $E[X_T] > E[X_0]$. For instance, if $a = b$ and $m = 2$, then you will always have $\$(X_0 + a)$ if you quit after the first win. However, for $a = b$ with $ap - bq < 0$, we have $p < q$ so that $q > 0.50$. Thus, $mq \geq 1$ when $m = 2$. Therein lies the paradox of the classic martingale strategy. Although, $E[X_T] > E[X_0]$ in this case, by Theorem 2, the average amount of your last bet will be $+\infty$.

## 7. The Maximum Guaranteed Number of Bets

Given an initial stake $\$X_0$, how many bets $n$ can we be guaranteed to make with the general martingale strategy? Of course, any win along the way will allow more bets to be made; but we must be able to cover the deficit $D_{n+1}$ accrued from $n$ initial losses in a row. So from Equation (7), we must have

$$X_0 \geq \sum_{i=1}^{n} m^{i-1} b = D_{n+1} = \begin{cases} nb & \text{if } m = 1 \\ b\left(\dfrac{m^n - 1}{m - 1}\right) & \text{if } m > 1. \end{cases} \tag{11}$$

Solving for $n$ gives us the maximum number of bets that we can be sure to cover:

$$n = \begin{cases} \lfloor X_0/b \rfloor & \text{if } m = 1 \\ \lfloor \ln\left(1 + X_0(m-1)/b\right) / \ln m \rfloor & \text{if } m > 1. \end{cases} \tag{12}$$

For example, with a \$3000 stake, an initial bet of \$20, and tripling the bet after each loss, then $\lfloor \ln(301) / \ln 3 \rfloor = 5$ bets definitely can be made, which exhaust \$2420 if all bets result in losses.

## 8. Choosing the Multiplier $m$

If we are always making the same bet of $\$b$, then by Equation (2) the best to hope for is to come out at least even with our initial stake of $\$X_0$. We then can sustain $\lfloor a/b \rfloor$ losses in a row and still break even with a win on the next bet. For instance, in American roulette, a square bet is a bet on a block of four numbers for which $p = 4/38$ and $q = 34/38$, with an 8:1 payoff ratio. Thus, we will still break even with 8 losses in a row followed by a win. And the chance of 9 losses in a row is only $(34/38)^9 \approx 0.3675$, so there is 63.25% chance of winning *within* 9 tries.

But how many wins $W_n$ are necessary to break even after any sequence of $n$ bets of $\$b$? The final fortune is now $X_n = X_0 + aW_n - b(n - W_n)$. In order to have $X_n \geq X_0$, we need $aW_n \geq b(n - W_n)$, which means that the number of wins $W_n$ must satisfy $W_n \geq nb/(a + b)$.

Here, $W_n$ is a binomial distribution where $P(W_n = k) = \binom{n}{k} p^k q^{n-k}$ for $0 \leq k \leq n$. For example, in a sequence of $n = 12$ roulette square bets, with $a = 8b$, then at least 2 wins are needed to break even. But this event only has probability $P(W_{12} \geq 2) = 1 - P(W_{12} = 0) - P(W_{12} = 1) = 1 - (34/38)^{12} - 12(4/38)(34/38)^{11} \approx 0.3651$. So with a square bet, we are much better off trying to win once within 9 wagers than trying to get 2 wins out of 12.

So with 9 or more losses on a square bet, we can no longer break even with just one win when using constant bets of \$$b$. The purpose of increasing the bets by a factor of $m > 1$ is to guarantee that we come out ahead after any win, no matter how many losses have occurred. But will any $m > 1$ guarantee that $X_T > X_0$?

From Equation (8), we see that we should choose $m \geq 1 + b/a$, which is equivalent to $a - b/(m-1) \geq 0$. If $m = 1 + b/a$, then the fortune after the first win will always be $X_T = X_0 + a$. And if $m > 1 + b/a$, then the overall gains are an increasing function of the number of plays needed for the first win. Indeed, because now $a - b/(m-1) > 0$ and $m > 1$, we have

$$\begin{aligned} X_T 1_{\{T\,=\,i\}} &= X_0 + \frac{b}{m-1} + \left(a - \frac{b}{m-1}\right) m^{i-1} \\ &< X_0 + \frac{b}{m-1} + \left(a - \frac{b}{m-1}\right) m^{i} \\ &= X_T 1_{\{T\,=\,i+1\}}. \end{aligned}$$

Equation (8) also shows that the overall *losses* are an increasing function of the number of plays needed for the first win when $1 \leq m < 1 + b/a$.

## 9. The Average Fortune Upon Stopping

With a finite initial stake $X_0$, we may not be able to keep betting until a win occurs. So suppose we decide in advance to make at most $n$ bets, but quit if we ever win. Then with $m \geq 1 + b/a$, we have $X_{T\wedge n} > X_0$ except if we lose all $n$ bets. We now shall derive the average final fortune $E[X_{T\wedge n}]$.

**Theorem 6.** *Let $T$ be the number of bets needed for the first win. Under the conditions of the general martingale strategy, the average fortune after the $(T \wedge n)$th bet is*

$$E[X_{T\wedge n}] = \begin{cases} E[X_0] + n(ap - bq) & \text{if } mq = 1 \\[2ex] E[X_0] + \dfrac{(ap - bq)(1 - (mq)^n)}{1 - mq} & \text{if } mq \neq 1. \end{cases}$$

*Proof.* We must adjust Equation (8) to account for a maximum of $n$ bets, and then subtract $D_{n+1}$, the total deficit accrued through $n$ initial losses, to account for $T > n$. If $q = 1$, then $X_{T\wedge n} = X_n = X_0 - D_{n+1}$, where $D_{n+1}$ is as in Equation (11), and the result follows.

For $q < 1$ and $m = 1$, we have

$$X_{T\wedge n} = X_0 + a1_{\{T\,\le\, n\}} - b\sum_{i\,=\,1}^{n}(i-1)1_{\{T\,=\,i\}} - nb\,1_{\{T\,>\,n\}}.$$

Using the fact that $\sum_{k=1}^{n} kx^{k-1} = (1-x^n - nx^n + nx^{n+1})/(1-x)^2$, for $x \neq 1$, we obtain the result in this case ($mq < 1$, $p = 1 - mq$) by

$$\begin{aligned}
E[X_{T\wedge n}] &= E[X_0] + (a+b)P(T \le n) - b\sum_{i\,=\,1}^{n} i\,P(T=i) - nbP(T>n)\\
&= E[X_0] + (a+b)(1-q^n) - bp\sum_{i\,=\,1}^{n} i\,q^{i-1} - nbq^n\\
&= E[X_0] + (a+b)(1-q^n) - bp\left(\frac{1-q^n-nq^n+nq^{n+1}}{(1-q)^2}\right) - nbq^n\\
&= E[X_0] + \left(\frac{ap-bq}{p}\right)(1-q^n).
\end{aligned}$$

For $q < 1$ but $m > 1$, we have

$$\begin{aligned}
X_{T\wedge n} \;&=\; X_0 + \left(\frac{b}{m-1}\right)1_{\{T\,\le\,n\}} + \sum_{i\,=\,1}^{n}\left(\left(a - \frac{b}{m-1}\right)m^{i-1}\right)1_{\{T\,=\,i\}}\\
&\quad - b\left(\frac{m^n-1}{m-1}\right)1_{\{T\,>\,n\}},
\end{aligned}$$

which gives

$$\begin{aligned}
E[X_{T\wedge n}] &= E[X_0] + \frac{b(1-q^n)}{m-1} + \left(a - \frac{b}{m-1}\right)p\sum_{i\,=\,0}^{n-1}(mq)^i\\
&\quad -b\left(\frac{m^n-1}{m-1}\right)q^n\\
&= \begin{cases} E[X_0] + n\left(ap - \frac{bp}{1/q-1}\right) & \text{if } mq = 1\\[2ex] E[X_0] + \frac{b(1-(mq)^n)}{m-1} + \left(ap - \frac{bp}{m-1}\right)\left(\frac{1-(mq)^n}{1-mq}\right) & \text{if } mq \neq 1\end{cases}\\
&= \begin{cases} E[X_0] + n(ap-bq) & \text{if } mq = 1\\[2ex] E[X_0] + \dfrac{(ap-bq)(1-(mq)^n)}{1-mq} & \text{if } mq \neq 1.\end{cases}
\end{aligned}$$

For $q < 1$ and $a < (q/p)b$ as in Theorem 5, we have $\lim_{n\to\infty} E[X_{T\wedge n}] = E[X_T]$ except when $m \geq 1 + b/a$ (which implies $mq > 1$). In that case, $\lim_{n\to\infty} E[X_{T\wedge n}] = -\infty$. We also see that $E[X_{T\wedge n}]$ for $mq = 1$, and more importantly, $E[X_{T\wedge n}] < E[X_0]$ when $a < (q/p)b$. □

## 10. The Desired Number of Bets

Finally, suppose we want to have a high probability $r$ of coming out ahead within $n$ bets. With $m \geq 1 + b/a$, we will come out ahead after the first win. So we want the probability of a win within $n$ bets to be at least $r$. That is, we want $P(T \leq n) = 1 - q^n \geq r$. Solving for $n$, for $q < 1$, the minimum number of bets that we must be able to make is

$$n = \left\lceil \frac{\ln(1-r)}{\ln q} \right\rceil. \tag{13}$$

Then to be able to make these $n$ bets with an initial bet of \$$b$, we must have an initial stake $X_0$ that satisfies Equation (11). Using $m \geq 1 + b/a$, we will then have at least probability $r$ of coming out ahead when quitting after the first win or after a total of these $n$ bets.

## 11. Comparison of Results

We shall illustrate our main results using a column bet in American roulette, for which $p = 12/38$, $q = 26/38$, with a 2:1 payoff ratio. Our initial bet will be $b = \$10$ (with $a = \$20$), and we wish to have $r = 0.95$ probability of coming out ahead. How many bets $n$ must we be able to make; what multiplier $m$ should we use; what initial stake $X_0$ do we need? When quitting after the first win or after $n$ bets, what is the average final fortune? We then shall compare the results to the scenarios of (i) making $n$ bets of \$$b$, and (ii) making bets of \$$b$, but quitting if we ever get ahead or after at most $n$ bets, which is the strategy discussed in [6].

From (13), we need to be able to make $n = \lceil \ln(0.05)/\ln(26/38) \rceil = 8$ bets, which actually gives us a $1 - (26/38)^8 \approx 0.952$ probability of winning within these 8 bets. With a 2:1 payoff, we need $m \geq 1 + 1/2 = 1.5$. We shall use $m = 1.6$ (and hence $mq > 1$). We note that by choosing $m < 2$, our bets will not increase dramatically and should stay within the table limits. Now from (11), we need an initial stake of at least $10(1.6^8 - 1)/0.6 \approx \$699.16$. (With $m = 2$, we would need $X_0 \geq \$2550$.)

Theorem 2 then gives the average amount of the last bet as $E[b_{T\wedge 8}] \approx \$48.32$. From Theorem 6, the average final fortune, with $X_0 = \$700$, is $E[X_{T\wedge 8}] \approx \$694.10$. In actual practice though, we would probably round the bets $b_i = 1.6^{i-1} \times 10$ up to the nearest dollar, which requires $X_0 = \$701$. These results, with adjusted averages, are given in the following table.

| **Outcome** | **Probability** | **Last Bet** | **Final Fortune** |
|---|---|---|---|
| $W$ | $p \approx 0.3158$ | \$10 | $701 + 20 = \$721$ |
| $LW$ | $qp \approx 0.2161$ | \$16 | $701 - 10 + 32 = \$723$ |
| $LLW$ | $q^2p \approx 0.1478$ | \$26 | $701 - 26 + 52 = \$727$ |
| $LLLW$ | $q^3p \approx 0.10115$ | \$41 | $701 - 52 + 82 = \$731$ |
| $LLLLW$ | $q^4p \approx 0.0692$ | \$66 | $701 - 93 + 132 = \$740$ |
| $LLLLLW$ | $q^5p \approx 0.04735$ | \$105 | $701 - 159 + 210 = \$752$ |
| $LLLLLLW$ | $q^6p \approx 0.0324$ | \$168 | $701 - 264 + 336 = \$773$ |
| $LLLLLLLW$ | $q^7p \approx 0.0222$ | \$269 | $701 - 432 + 538 = \$807$ |
| $LLLLLLLL$ | $q^8 \approx 0.0480$ | \$269 | $701 - 701 = \$0$ |

$$\mathbf{X_0 = \$701,\ m \approx 1.6,\ p = 12/38,\ q = 26/38,\ n = 8,}$$
$$\mathbf{E[b_{T\wedge 8}] \approx \$48.47,\ P(X_{T\wedge 8} > 701) \approx 0.952,\ E[X_{T\wedge 8}] \approx \$695.07.}$$

Following are the results of two other strategies:
(i) For sequences of 8 bets of \$10 each with $X_0 = \$701$, Equation (2) gives the average final fortune as $E[X_8] \approx \$696.79$, which is higher than $E[X_{T\wedge 8}]$ using $m \approx 1.6$. However now we require at least $\lceil nb/(a+b) \rceil = 3$ wins to come out ahead, which occurs only with probability 0.4881. The following table shows the possible outcomes.

| **Outcome** | **Probability of k Wins** | **Final Fortune** |
|---|---|---|
| (any order) | $\binom{8}{k} p^k q^{8-k}$ | $701 + 20k - 10(8-k)$ |
| $8W$ | 0.0001 | \$861 |
| $7W,\ 1L$ | 0.0017 | \$831 |
| $6W,\ 2L$ | 0.0130 | \$801 |
| $5W,\ 3L$ | 0.0563 | \$771 |
| $4W,\ 4L$ | 0.1526 | \$741 |
| $3W,\ 5L$ | 0.2644 | \$711 |
| $2W,\ 6L$ | 0.2865 | \$681 |
| $1W,\ 7L$ | 0.1773 | \$651 |
| $8L$ | 0.0480 | \$621 |

$$\mathbf{P(X_8 > 701) \approx 0.4881,\ E[X_8] \approx \$696.79.}$$

(ii) Suppose now that we make at most 8 bets of \$10 each, but quit if the fortune $X_i$ ever surpasses $X_0 = \$701$. By scaling down the values, we can compute $E[X_{T\wedge 8}]$ and $P(X_{T\wedge 8} > 701)$ using matrix products. We first divide all values by $\gcd(a,b)$, which is 10 in this case. Now $X_0 = \$70.1$, $b = 1$, and $a = 2$. After 8 bets, the minimum possible fortune is \$62.1; thus we subtract 62.1 from each fortune value. We now have a Markov chain that begins at height $X_0 = 8$, moves up 2 units at a time with probability $p = 12/38$ or down 1 unit at a time with probability $q = 1 - p$, and stops after at most 8 steps or if ever reaching an upper boundary of 9 or 10.

We then let $A$ be the $1 \times 11$ matrix of possible position heights, let $B$ be the $1 \times 11$ initial probability state matrix, and let $C$ be the $11 \times 11$

matrix of transition probabilities, where $c_{i\,j} = P(X_{k+1} = j \mid X_k = i)$, for $0 \le i,\ j \le 10$ and all $k \ge 0$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $B \times C^8$ gives the probability state of being at height $i$ upon stopping or at most 8 steps. These probabilities coincide with the actual fortune being $10 \times (i + 62.1)$. $E[X_{T \wedge 8}] = 10 \times (B \times C^8 \times A^T + 62.1) \approx \$698.84$. The following table shows the possible outcomes.

| **Final Fortune** | **Probability** |
|---|---|
| \$721 | 0.3832 |
| \$711 | 0.3130 |
| \$681 | 0.1228 |
| \$651 | 0.1330 |
| \$621 | 0.0480 |

$$\mathbf{P(X_{T\wedge 8} > 701) \approx 0.6962,\ E[X_{T\wedge 8}] \approx \$698.84}.$$

Because of the one case of losing all bets and thus losing the whole stake, the martingale strategy yields the smallest average fortune. However, it provides the highest probability of coming out ahead, which also can be chosen in advance. In our example, you will come out ahead about 19 out of 20 times, with manageable bets, provided you are willing to risk it all.

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