SOME APPLICATIONS OF THE FUNDAMENTAL THEOREM OF HERMITIAN K-THEORY

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Abstract. In this work we show how to use the Karoubi's fundamental theorem of Hermitian K-theory [6] to prove some results in L-Theory using these same results in algebraic K-Theory.

1. Introduction

Let I be a ring (eventually without unit). I is excisive for the algebraic (resp. Hermitian) K-Theory, if for every Cartesian diagram of unitary (resp. Hermitian) rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ A_2 & \longrightarrow & A' \end{array}$$

such that $I \simeq \ker \varphi_1$ and φ_1 is surjective, we have

$$K_n(\varphi_1) \simeq K_n(\varphi_2) \text{ (resp. } {}_\varepsilon L_n(\varphi_1) \simeq {}_\varepsilon L_n(\varphi_2)) \text{ for every } n \in \mathbb{Z}.$$

In particular, we have the following long exact sequence:

$$\cdots \longrightarrow K_{n+1}(A') \longrightarrow K_n(A) \longrightarrow K_n(A_1) \oplus K_n(A_2) \longrightarrow K_n(A') \longrightarrow K_{n-1}(A) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow {}_\varepsilon L_{n+1}(A') \longrightarrow {}_\varepsilon L_n(A) \longrightarrow {}_\varepsilon L_n(A_1) \oplus {}_\varepsilon L_n(A_2) \longrightarrow {}_\varepsilon L_n(A') \longrightarrow {}_\varepsilon L_{n-1}(A) \longrightarrow \cdots .$$

As examples of excisive rings for the algebraic K-Theory, we can give $\mathbf{C}^*$-algebras and H-unital algebras [12]. In the first part of this work, we use the Karoubi's fundamental theorem of Hermitian K-Theory, to prove that if a ring is excisive for the algebraic K-Theory, then it is excisive for the Hermitian K-Theory. We also prove the same result for the K-Theory with coefficients in $\mathbb{Z}/q$.

Let A be an involutive Banach algebra. The canonical maps:

A part of this work was the subject of a note in CRAS (see [2]).

$$BGL\,(A) \xrightarrow{\sigma} BGL^{top}\,(A) \text{ and } B_{\varepsilon}O\,(A) \xrightarrow{\tau} B_{\varepsilon}O^{top}\,(A)$$

induce the following homomorphisms

$$K_n\,(A) \xrightarrow{\sigma_n} K_n^{top}\,(A) \text{ and } {}_{\varepsilon}L_n\,(A) \xrightarrow{\tau_n} {}_{\varepsilon}L_n^{top}\,(A).$$

In the second part of this work, using the fundamental theorem of Hermitian K-Theory, we prove that if σ_n is an isomorphism for all $n \geq 0$, then the same is true for τ_n. Since Wodzicki and Suslin have shown that for stable $\mathbf{C}^*$-algebras, the σ_n are isomorphisms for all $n \geq 0$ [12], then for these algebras, algebraic and Hermitian K-Theory groups coincide.

2. Review of Known Facts

2.1. Here we recall some results obtained by using the algebraic suspension SA of a ring A. (see [10], p. 327, for the definition of the algebraic suspension).

Theorem 2.1.1. [13] *Let A be a unitary ring. We have natural homotopy equivalence*

$$\Omega BGL\,(SA)^{+} \sim K_0\,(A) \times BGL\,(A)^{+}.$$

The group $K_0\,(A)$ is endowed with the discrete topology. In particular, for every $n \geq 1$, we have

$$K_n\,(SA) \simeq K_{n-1}\,(A).$$

Theorem 2.1.2. [5] *Let A be a Hermitian ring. We have natural homotopy equivalence*

$$\Omega B_{\varepsilon}O\,(SA)^{+} \sim {}_{\varepsilon}L_0\,(A) \times B_{\varepsilon}O\,(A)^{+}\ .$$

The group ${}_{\varepsilon}L_0\,(A)$ is endowed with the discrete topology. In particular, for every $n \geq 1$, we have

$${}_{\varepsilon}L_n\,(SA) \simeq {}_{\varepsilon}L_{n-1}\,(A).$$

These theorems are used to define groups K_n and ${}_{\varepsilon}L_n$ for all $n < 0$. For a unitary ring (resp. Hermitian ring) A and $n < 0$, we set

$$K_n\,(A) = K_0\,(S^{-n}A) \text{ (resp. } {}_{\varepsilon}L_n\,(A) = {}_{\varepsilon}L_0\,(S^{-n}A)\text{).}$$

2.2. Let A be a Hermitian ring. The hyperbolic functor [4] induces a group homomorphism

$$K_0\,(A) \longrightarrow {}_{\varepsilon}L_0\,(A)$$

and the homomorphisms

$$GL_r\,(A) \longrightarrow {}_{\varepsilon}O_{r,r}\,(A)$$

defined by the following correspondence

$$M \longrightarrow \begin{pmatrix} M & 0 \\ 0 & {}^{t}\overline{M}^{-1} \end{pmatrix}$$

induces a map

$$BGL\,(A)^{+} \longrightarrow B_{\varepsilon}O\,(A)^{+}.$$

We denote ${}_{\varepsilon}\mathcal{U}\,(A)$ as the homotopic fiber of the map

$$K_0\,(A) \times BGL\,(A)^{+} \longrightarrow {}_{\varepsilon}L_0\,(A) \times B_{\varepsilon}O\,(A)^{+}.$$

Similarly, the forgetful functor [4] induces a group homomorphism

$${}_{\varepsilon}L_0\,(A) \longrightarrow K_0\,(A)$$

and the natural inclusions

$${}_{\varepsilon}O_{r,r}\,(A) \longrightarrow GL_{2r}\,(A)$$

induce a map

$$B_{\varepsilon}O^{+}\,(A) \longrightarrow BGL\,(A)^{+}.$$

We denote ${}_{\varepsilon}\mathcal{V}\,(A)$ as the homotopic fiber of the map

$${}_{\varepsilon}L_0\,(A) \times B_{\varepsilon}O\,(A)^{+} \longrightarrow K_0\,(A) \times BGL\,(A)^{+}.$$

Theorem 2.2.1. [6] *Let A be a Hermitian ring containing in its center an element λ, such that $\lambda + \overline{\lambda} = 1$. (This condition is satisfied if, for example, 2 is invertible in A). Then there exists a natural homotopy equivalence between spaces $\Omega_{\varepsilon}\mathcal{U}\,(A)$ and ${}_{-\varepsilon}\mathcal{V}\,(A)$.*

We recall that the topological version of this theorem induces Bott periodicity in the real and complex cases. This interpretation of the Bott periodicity doesn't use Clifford algebras [4].

For $n \geq 0$, we let

$${}_{\varepsilon}U_n\,(A) = \pi_n\,({}_{\varepsilon}\mathcal{U}\,(A)) \text{ and } {}_{\varepsilon}V_n\,(A) = \pi_n\,({}_{\varepsilon}\mathcal{V}\,(A))$$

and for $n < 0$, we let

$${}_{\varepsilon}U_n\,(A) = {}_{\varepsilon}U_0\,(S^{-n}A) \text{ and } {}_{\varepsilon}V_n\,(A) = {}_{\varepsilon}V_0\,(S^{-n}A).$$

For every $n \in \mathbb{Z}$, we have

$${}_{\varepsilon}U_{n+1}\,(A) \simeq {}_{-\varepsilon}V_n\,(A).$$

We also have the following long exact sequences

$$\cdots \longrightarrow K_{n+1}\,(A) \longrightarrow {}_{\varepsilon}V_n\,(A) \longrightarrow {}_{\varepsilon}L_n\,(A) \longrightarrow K_n\,(A) \longrightarrow {}_{\varepsilon}V_{n-1}\,(A) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow {}_{\varepsilon}L_{n+1}\,(A) \longrightarrow {}_{\varepsilon}U_n\,(A) \longrightarrow K_n\,(A) \longrightarrow {}_{\varepsilon}L_n\,(A) \longrightarrow {}_{\varepsilon}U_{n-1}\,(A) \longrightarrow \cdots.$$

2.3. Let A be a unitary (resp. Hermitian) ring. The space $K_0\,(A) \times BGL\,(A)^{+}$ (resp. ${}_{\varepsilon}L_0\,(A) \times B_{\varepsilon}O\,(A)^{+}$) will be denoted $\mathcal{K}\,(A)$ (resp. ${}_{\varepsilon}\mathcal{L}\,(A)$). Let f be a homomorphism of unitary (resp. Hermitian) rings

$$f\colon A \longrightarrow B.$$

We will recall a construction, due mainly to Wagoner [13], of the groups $K_n\,(f)$ (resp. ${}_{\varepsilon}L_n\,(f)$). Let $\Gamma\,(f)$ be the fibered product of SA and CB over SB:

$$\begin{array}{ccc} \Gamma(f) & \longrightarrow & CB \\ \downarrow & & \downarrow \\ SA & \longrightarrow & SB \end{array} .$$

The space $\Omega\mathcal{K}(\Gamma(f))$ (resp. $\Omega_{\varepsilon}\mathcal{L}(\Gamma(f))$) has the same homotopy type as the homotopic fiber $\mathcal{K}(f)$ (resp. ${}_{\varepsilon}\mathcal{L}(f)$) of the map

$$\mathcal{K}(A) \longrightarrow \mathcal{K}(B) \text{ (resp. } {}_{\varepsilon}\mathcal{L}(A) \longrightarrow {}_{\varepsilon}\mathcal{L}(B)).$$

For every $n \geq 0$, we let

$$K_n(f) = \pi_n(\mathcal{K}(f)) \text{ (resp. } {}_{\varepsilon}L_n(f) = \pi_n({}_{\varepsilon}\mathcal{L}(f)))$$

and for $n < 0$, we let

$$K_n(f) = K_0(S^{-n}f) \text{ (resp. } {}_{\varepsilon}L_n(f) = {}_{\varepsilon}L_0(S^{-n}f)).$$

So for all $n \in \mathbb{Z}$, we have

$$K_n(Sf) \simeq K_{n-1}(f) \text{ (resp. } {}_{\varepsilon}L_n(Sf) \simeq {}_{\varepsilon}L_{n-1}(f))$$

and

$$K_n(f) = K_{n+1}(\Gamma(f)) \text{ (resp. } {}_{\varepsilon}L_n(f) \simeq {}_{\varepsilon}L_{n+1}(\Gamma(f))).$$

We also have the following long exact sequences

$$\cdots \longrightarrow K_{n+1}(B) \longrightarrow K_n(f) \longrightarrow K_n(A) \longrightarrow K_n(B) \longrightarrow K_{n-1}(f) \longrightarrow \cdots$$

$$\cdots \longrightarrow {}_{\varepsilon}L_{n+1}(B) \longrightarrow {}_{\varepsilon}L_n(f) \longrightarrow {}_{\varepsilon}L_n(A) \longrightarrow {}_{\varepsilon}L_n(B) \longrightarrow {}_{\varepsilon}L_{n-1}(f) \longrightarrow \cdots$$

2.4. Excision in K-Theory.

Definition 2.4.1. *We say that a diagram of unitary (resp. Hermitian) rings*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \varphi_2 \downarrow & & \downarrow \varphi_1 \\ A_2 & \longrightarrow & A' \end{array}$$

is excisive for the algebraic (resp. Hermitian) K-Theory, if for every $n \in \mathbb{Z}$, we have

$$K_n(\varphi_1) \simeq K_n(\varphi_2), \text{ resp. } {}_{\varepsilon}L_n(\varphi_1) \simeq {}_{\varepsilon}L_n(\varphi_2).$$

For an excisive diagram for the algebraic (resp. Hermitian) K-Theory, in particular, we have the Mayer-Vietoris long exact sequence

$$\cdots \longrightarrow K_{n+1}(A') \longrightarrow K_n(A) \longrightarrow K_n(A_1) \oplus K_n(A_2) \longrightarrow K_n(A') \longrightarrow K_{n-1}(A) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow {}_{\varepsilon}L_{n+1}(A') \longrightarrow {}_{\varepsilon}L_n(A) \longrightarrow {}_{\varepsilon}L_n(A_1) \oplus {}_{\varepsilon}L_n(A_2) \longrightarrow {}_{\varepsilon}L_n(A') \longrightarrow {}_{\varepsilon}L_{n-1}(A) \longrightarrow \cdots .$$

Definition 2.4.2. *Let I be a ring (eventually without unit). We say that I is excisive for the algebraic (resp. Hermitian) K-Theory, if every Cartesian diagram of unitary (resp. Hermitian) rings*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ {\scriptstyle \varphi_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_1} \\ A_2 & \longrightarrow & A' \end{array}$$

such that $I \simeq \ker \varphi_1$ and φ_1 is surjective, is excisive for the algebraic (resp. Hermitian) K-Theory.

Remark 2.4.3. *Given a diagram of unitary (resp. Hermitian) rings*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ {\scriptstyle \varphi_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_1} \\ A_2 & \longrightarrow & A' \end{array}$$

such that φ_1 is surjective. Then we have

$$K_0(\varphi_1) \simeq K_0(\varphi_2) \ [1].$$

Respectively,

$${}_{\varepsilon}L_0(\varphi_1) \simeq {}_{\varepsilon}L_0(\varphi_2) \ [8].$$

Note that using Proposition 2.5 of [6, *p.* 269] we show that this definition of the relative groups ${}_{\varepsilon}L_0(\varphi_1)$ and ${}_{\varepsilon}L_0(\varphi_2)$ coincide with that of [9].

2.5. **Examples of Excisive Rings for the Algebraic K-Theory.**

Theorem 2.5.1. [12] *Every $\mathbb{C}^*$-algebra is excisive for the algebraic K-Theory.*

Let A be a $\mathbb{Q}$-algebra. We say that A is H-unital if the complex

$$\cdots \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} A^{\otimes n-1} \xrightarrow{b'} \cdots \xrightarrow{b'} A \otimes A \xrightarrow{b'} A$$

is acyclic. For every $n \geq 2$, the homomorphism b' is defined on $A^{\otimes n}$ by the following formula

$$b'(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \textstyle\sum_{i=2}^{n} (-1)^i \, a_1 \otimes a_2 \otimes a_{i-1}a_i \otimes \cdots \otimes a_n.$$

Theorem 2.5.2. [12] *Every H-unital ring is excisive for the algebraic K-Theory.*

3

In the following, we suppose that 2 is invertible in the considered rings.

3.1.

Theorem 3.1.1. *Given $p \in \mathbb{Z}$ and a Cartesian diagram of Hermitian rings*

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ {\scriptstyle \varphi_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_1} \\ A_2 & \longrightarrow & A' \end{array}$$

such that

$$K_n(\varphi_1) \simeq K_n(\varphi_2) \text{ for every } n \geq p,\ {}_{\varepsilon}L_p(\varphi_1) \simeq {}_{\varepsilon}L_p(\varphi_2) \text{ and } {}_{\varepsilon}L_{p+1}(\varphi_1) \simeq {}_{\varepsilon}L_{p+1}(\varphi_2).$$

Then

$${}_{\varepsilon}L_n(\varphi_1) \simeq {}_{\varepsilon}L_n(\varphi_2) \text{ for all } n \geq p.$$

3.2. Before proving this theorem, we will define for a homomorphism of Hermitian rings

$$f : A \longrightarrow B$$

and for every $n \in \mathbb{Z}$, relative groups ${}_{\varepsilon}U_n(f)$ and ${}_{\varepsilon}V_n(f)$. Let $f : A \longrightarrow B$ be a homomorphism of Hermitian rings. We have the following commutative diagrams

$$\begin{array}{ccccc} {}_{\varepsilon}\mathcal{U}(f) & \longrightarrow & \mathcal{K}(f) & \longrightarrow & {}_{\varepsilon}\mathcal{L}(f) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ {}_{\varepsilon}\mathcal{U}(A) & \longrightarrow & \mathcal{K}(A) & \longrightarrow & {}_{\varepsilon}\mathcal{L}(A) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ {}_{\varepsilon}\mathcal{U}(B) & \longrightarrow & \mathcal{K}(B) & \longrightarrow & {}_{\varepsilon}\mathcal{L}(B). \end{array}$$

The fiber of the map $\mathcal{K}(f) \longrightarrow {}_{\varepsilon}\mathcal{L}(f)$ is equal to the fiber of the map ${}_{\varepsilon}\mathcal{U}(A) \longrightarrow {}_{\varepsilon}\mathcal{U}(B)$. We denote ${}_{\varepsilon}\mathcal{U}(f)$ as this common fiber. We also have the following commutative diagrams

$$\begin{array}{ccccc} {}_{\varepsilon}\mathcal{V}(f) & \longrightarrow & {}_{\varepsilon}\mathcal{L}(f) & \longrightarrow & \mathcal{K}(f) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ {}_{\varepsilon}\mathcal{V}(A) & \longrightarrow & {}_{\varepsilon}\mathcal{L}(A) & \longrightarrow & \mathcal{K}(A) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ {}_{\varepsilon}\mathcal{V}(B) & \longrightarrow & {}_{\varepsilon}\mathcal{L}(B) & \longrightarrow & \mathcal{K}(B). \end{array}$$

The fiber of the map ${}_{\varepsilon}\mathcal{L}(f) \longrightarrow \mathcal{K}(f)$ is equal to the fiber of the map ${}_{\varepsilon}\mathcal{V}(A) \longrightarrow {}_{\varepsilon}\mathcal{V}(B)$. We denote ${}_{\varepsilon}\mathcal{V}(f)$ as this common fiber. For every $n \geq 0$, we let

$${}_{\varepsilon}U_n(f) = \pi_n({}_{\varepsilon}\mathcal{U}(f)) \text{ and } {}_{\varepsilon}V_n(f) = \pi_n({}_{\varepsilon}\mathcal{V}(f)).$$

For $n < 0$, we let

$$_{\varepsilon}U_n(f) = {_{\varepsilon}U_0}(S^{-n}f) \text{ and } _{\varepsilon}V_n(f) = {_{\varepsilon}V_0}(S^{-n}f).$$

For every $n \in \mathbb{Z}$, we have the following long exact sequences

$$\cdots \longrightarrow {_{\varepsilon}U_{n+1}}(B) \longrightarrow {_{\varepsilon}U_n}(f) \longrightarrow {_{\varepsilon}U_n}(A) \longrightarrow {_{\varepsilon}U_n}(B) \longrightarrow {_{\varepsilon}U_{n-1}}(f) \longrightarrow \cdots$$
$$\cdots \longrightarrow {_{\varepsilon}V_{n+1}}(B) \longrightarrow {_{\varepsilon}V_n}(f) \longrightarrow {_{\varepsilon}V_n}(A) \longrightarrow {_{\varepsilon}V_n}(B) \longrightarrow {_{\varepsilon}V_{n-1}}(f) \longrightarrow \cdots$$
$$\cdots \longrightarrow K_{n+1}(f) \longrightarrow {_{\varepsilon}V_n}(f) \longrightarrow {_{\varepsilon}L_n}(f) \longrightarrow K_n(f) \longrightarrow {_{\varepsilon}V_{n-1}}(f) \longrightarrow \cdots$$
$$\cdots \longrightarrow {_{\varepsilon}L_{n+1}}(f) \longrightarrow {_{\varepsilon}U_n}(f) \longrightarrow K_n(f) \longrightarrow {_{\varepsilon}L_n}(f) \longrightarrow {_{\varepsilon}U_{n-1}}(f) \longrightarrow \cdots.$$

For the proof of Theorem 3.1.1 we will need the following lemma.

3.3.

Lemma 3.3.1. *Let $f : A \longrightarrow B$ be a homomorphism of Hermitian rings. For every $n \in \mathbb{Z}$, we have*

$$_{\varepsilon}U_{n+1}(f) \simeq {_{-\varepsilon}V_n}(f).$$

3.4.

Proof. Knowing that for any Hermitian ring D the homotopy equivalence

$$\Omega_{\varepsilon}\mathcal{U}(D) \sim {_{-\varepsilon}\mathcal{V}}(D)$$

is natural, we have the following commutative diagrams

$$\begin{array}{ccccc}
{_{-\varepsilon}\mathcal{V}}(f) & \longrightarrow & {_{-\varepsilon}\mathcal{V}}(A) & \longrightarrow & {_{-\varepsilon}\mathcal{V}}(B) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_{\varepsilon}\mathcal{U}(f) & \longrightarrow & \Omega_{\varepsilon}\mathcal{U}(A) & \longrightarrow & \Omega_{\varepsilon}\mathcal{U}(B).
\end{array}$$

Then for all $n \in \mathbb{Z}$, we have the following diagrams of long exact sequences

$$\begin{array}{ccccccccc}
{_{-\varepsilon}V_{n+1}}(A) & \longrightarrow & {_{-\varepsilon}V_{n+1}}(B) & \longrightarrow & {_{-\varepsilon}V_n}(f) & \longrightarrow & {_{-\varepsilon}V_n}(A) & \longrightarrow & {_{-\varepsilon}V_n}(B) \\
\wr\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow \\
{_{\varepsilon}U_{n+2}}(A) & \longrightarrow & {_{\varepsilon}U_{n+2}}(B) & \longrightarrow & {_{\varepsilon}U_{n+1}}(f) & \longrightarrow & {_{\varepsilon}U_{n+1}}(A) & \longrightarrow & {_{\varepsilon}U_{n+1}}(B).
\end{array}$$

Hence, we have proved the lemma. □

3.5. **Proof of Theorem 3.1.1.** For all $n \in \mathbb{Z}$, the homomorphism $A \longrightarrow A_1$ induces the following diagrams of long exact sequences

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & {}_{\varepsilon}L_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}U_n(\varphi_2) & \longrightarrow & K_n(\varphi_2) & \longrightarrow & {}_{\varepsilon}L_n(\varphi_2) & \longrightarrow & {}_{\varepsilon}U_{n-1}(\varphi_2) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & {}_{\varepsilon}L_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}U_n(\varphi_1) & \longrightarrow & K_n(\varphi_1) & \longrightarrow & {}_{\varepsilon}L_n(\varphi_1) & \longrightarrow & {}_{\varepsilon}U_{n-1}(\varphi_1) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & K_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}V_n(\varphi_2) & \longrightarrow & {}_{\varepsilon}L_n(\varphi_2) & \longrightarrow & K_n(\varphi_2) & \longrightarrow & {}_{\varepsilon}V_{n-1}(\varphi_2) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}V_n(\varphi_1) & \longrightarrow & {}_{\varepsilon}L_n(\varphi_1) & \longrightarrow & K_n(\varphi_1) & \longrightarrow & {}_{\varepsilon}V_{n-1}(\varphi_1) & \longrightarrow & \cdots .
\end{array}$$

Consider the following diagram of exact sequences

$$\begin{array}{ccccccccc}
{}_{\varepsilon}L_{p+1}(\varphi_2) & \longrightarrow & K_{p+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}V_p(\varphi_2) & \longrightarrow & {}_{\varepsilon}L_p(\varphi_2) & \longrightarrow & K_p(\varphi_2) \\
\wr\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow \\
{}_{\varepsilon}L_{p+1}(\varphi_1) & \longrightarrow & K_{p+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}V_p(\varphi_1) & \longrightarrow & {}_{\varepsilon}L_p(\varphi_1) & \longrightarrow & K_p(\varphi_1) .
\end{array}$$

We deduce that for any ε

$$ {}_{\varepsilon}V_p(\varphi_2) \simeq {}_{\varepsilon}V_p(\varphi_1). $$

Then we have

$$ {}_{\varepsilon}U_{p+1}(\varphi_2) \simeq {}_{\varepsilon}U_{p+1}(\varphi_1). $$

We proceed now by induction on n. Assume that

$$ {}_{\varepsilon}L_n(\varphi_2) \simeq_{\varepsilon} L_n(\varphi_1) \text{ and } {}_{\varepsilon}U_n(\varphi_2) \simeq {}_{\varepsilon}U_n(\varphi_1). $$

The diagram of exact sequences

$$\begin{array}{ccccccc}
K_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}L_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}U_n(\varphi_2) & \longrightarrow & K_n(\varphi_2) \\
\wr\downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow \\
K_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}L_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}U_n(\varphi_1) & \longrightarrow & K_n(\varphi_1)
\end{array}$$

prove that the homomorphism

$$ {}_{\varepsilon}L_{n+1}(\varphi_2) \longrightarrow_{\varepsilon} L_{n+1}(\varphi_1) $$

is surjective. Consider the following diagram

$$\begin{array}{ccccccccc}
{}_{\varepsilon}L_{n+1}(\varphi_2) & \longrightarrow & K_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}V_n(\varphi_2) & \longrightarrow & {}_{\varepsilon}L_n(\varphi_2) & \longrightarrow & K_n(\varphi_2) \\
\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow \\
{}_{\varepsilon}L_{n+1}(\varphi_1) & \longrightarrow & K_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}V_n(\varphi_1) & \longrightarrow & {}_{\varepsilon}L_n(\varphi_1) & \longrightarrow & K_n(\varphi_1) .
\end{array}$$

We deduce that for any ε

$$ {}_{\varepsilon}V_n(\varphi_2) \simeq {}_{\varepsilon}V_n(\varphi_1). $$

Consequently, we have

$$ {}_{\varepsilon}U_{n+1}(\varphi_2) \simeq {}_{\varepsilon}U_{n+1}(\varphi_1). $$

Finally, consider the diagram of exact sequences

$$\begin{array}{ccccccccc}
{}_{\varepsilon}U_{n+1}(\varphi_2) & \longrightarrow & K_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}L_{n+1}(\varphi_2) & \longrightarrow & {}_{\varepsilon}U_{n}(\varphi_2) & \longrightarrow & K_{n}(\varphi_2) \\
\wr\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow \\
{}_{\varepsilon}U_{n+1}(\varphi_1) & \longrightarrow & K_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}L_{n+1}(\varphi_1) & \longrightarrow & {}_{\varepsilon}U_{n}(\varphi_1) & \longrightarrow & K_{n}(\varphi_1).
\end{array}$$

It follows that

$${}_{\varepsilon}L_{n+1}(\varphi_2) \simeq {}_{\varepsilon}L_{n+1}(\varphi_1).$$

Hence, we have the following corollary.

Corollary 3.5.1. *Let I be a Hermitian ring. If I is excisive for the algebraic K-Theory, then it is also excisive for the Hermitian K-Theory.*

3.6.

Proof. Let I be an excisive ring for the algebraic K-Theory and consider the following Cartesian diagram of Hermitian rings

$$\begin{array}{ccc}
A & \longrightarrow & A_1 \\
{\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\
A_2 & \longrightarrow & A'
\end{array}$$

such that $I \simeq \ker\varphi_1$ and φ_1 is surjective. According to Remark 2.4.3 we have

$${}_{\varepsilon}L_0(\varphi_2) \simeq {}_{\varepsilon}L_0(\varphi_1).$$

The suspension of this diagram is also a Cartesian diagram and $S\varphi_1$ is also surjective. Then, we have (according to the Remark 2.4.3)

$${}_{\varepsilon}L_0(S\varphi_2) \simeq {}_{\varepsilon}L_0(S\varphi_1).$$

So we have

$${}_{\varepsilon}L_{-1}(\varphi_2) = {}_{\varepsilon}L_0(S\varphi_2) \simeq {}_{\varepsilon}L_0(S\varphi_1) = {}_{\varepsilon}L_{-1}(\varphi_1).$$

Then by Theorem 3.1.1 and for all $n \geq -1$, we have

$${}_{\varepsilon}L_n(\varphi_2) \simeq {}_{\varepsilon}L_n(\varphi_1).$$

For $n < -1$, we have

$${}_{\varepsilon}L_n(\varphi_2) = {}_{\varepsilon}L_0(S^{-n}\varphi_2) \simeq {}_{\varepsilon}L_0(S^{-n}\varphi_1) = {}_{\varepsilon}L_n(\varphi_1).$$

Hence, we have proved the corollary. □

4. Excision in Hermitian K-Theory with Coefficients in $\mathbb{Z}/q$

4.1. Let X be a topological space. For any $n \geq 2$, $\pi_n(X;\mathbb{Z}/q)$ will denote the nth homotopy group of X with coefficient in $\mathbb{Z}/q$ [11].

Definition 4.1.1. *Let A be a unitary (resp. Hermitian) ring. For all $n \geq 2$, we let*

$$K_n(A;\mathbb{Z}/q) = \pi_n(BGL(A)^+;\mathbb{Z}/q) \text{ resp. } {}_\varepsilon L_n(A;\mathbb{Z}/q) = \pi_n(B_\varepsilon O(A)^+;\mathbb{Z}/q).$$

For $n < 2$, we let

$$K_n(A;\mathbb{Z}/q) = K_2(S^{2-n}A;\mathbb{Z}/q), \text{ resp. } {}_\varepsilon L_n(A;\mathbb{Z}/q) = {}_\varepsilon L_2(S^{2-n}A;\mathbb{Z}/q).$$

Definition 4.1.2. *Let $f : A \longrightarrow B$ be a homomorphism of unitary (resp. Hermitian) rings. For all $n \geq 2$, we let*

$$K_n(f;\mathbb{Z}/q) = \pi_n(\mathcal{K}(f);\mathbb{Z}/q), \text{ resp. } {}_\varepsilon L_n(f;\mathbb{Z}/q) = \pi_n({}_\varepsilon\mathcal{L}(f);\mathbb{Z}/q).$$

For $n < 2$, we let

$$K_n(f;\mathbb{Z}/q) = K_2(S^{2-n}f;\mathbb{Z}/q), \text{ resp. } {}_\varepsilon L_n(f;\mathbb{Z}/q) = {}_\varepsilon L_2(S^{2-n}f;\mathbb{Z}/q).$$

For all $n \in \mathbb{Z}$, we have the following long exact sequences

$$\cdots \longrightarrow K_{n+1}(B;\mathbb{Z}/q) \longrightarrow K_n(f;\mathbb{Z}/q) \longrightarrow K_n(A;\mathbb{Z}/q) \longrightarrow K_n(B;\mathbb{Z}/q) \longrightarrow K_{n-1}(f;\mathbb{Z}/q) \longrightarrow \cdots$$

$$\cdots \longrightarrow {}_\varepsilon L_{n+1}(B;\mathbb{Z}/q) \longrightarrow {}_\varepsilon L_n(f;\mathbb{Z}/q) \longrightarrow {}_\varepsilon L_n(A;\mathbb{Z}/q) \longrightarrow {}_\varepsilon L_n(B;\mathbb{Z}/q) \longrightarrow {}_\varepsilon L_{n-1}(f;\mathbb{Z}/q) \longrightarrow \cdots .$$

Definition 4.1.3. *Let A be a Hermitian ring. For all $n \geq 2$, we let*

$${}_\varepsilon U_n(A;\mathbb{Z}/q) = \pi_n({}_\varepsilon\mathcal{U}(A);\mathbb{Z}/q), \text{ resp. } {}_\varepsilon V_n(A;\mathbb{Z}/q) = \pi_n({}_\varepsilon\mathcal{V}(A);\mathbb{Z}/q).$$

For $n < 2$, we let

$${}_\varepsilon U_n(A;\mathbb{Z}/q) = {}_\varepsilon U_2(S^{2-n}A;\mathbb{Z}/q), \text{ resp. } {}_\varepsilon V_n(A;\mathbb{Z}/q) = {}_\varepsilon V_2(S^{2-n}A;\mathbb{Z}/q).$$

Note that for all $n \in \mathbf{Z}$, we have

$${}_\varepsilon U_n(SA;\mathbb{Z}/q) \simeq {}_\varepsilon U_{n-1}(A;\mathbb{Z}/q),\ {}_\varepsilon V_n(SA;\mathbb{Z}/q) \simeq {}_\varepsilon V_{n-1}(A;\mathbb{Z}/q)$$

and

$${}_\varepsilon U_{n+1}(SA;\mathbb{Z}/q) \simeq {}_{-\varepsilon} V_n(A;\mathbb{Z}/q).$$

Definition 4.1.4. *Let $f : A \longrightarrow B$ be a homomorphism of Hermitian rings. For all $n \geq 2$, we let*

$${}_\varepsilon U_n(f;\mathbb{Z}/q) = \pi_n({}_\varepsilon\mathcal{U}(f);\mathbb{Z}/q), \text{ resp. } {}_\varepsilon V_n(f;\mathbb{Z}/q) = \pi_n({}_\varepsilon\mathcal{V}(f);\mathbb{Z}/q).$$

For $n < 2$, we let

$${}_\varepsilon U_n(f;\mathbb{Z}/q) = {}_\varepsilon U_2(S^{2-n}f;\mathbb{Z}/q), \text{ resp. } {}_\varepsilon V_n(f;\mathbb{Z}/q) = {}_\varepsilon V_2(S^{2-n}f;\mathbb{Z}/q).$$

To simplify the writing, groups $K_n(.;\mathbb{Z}/q)$, ${}_\varepsilon L_n(.;\mathbb{Z}/q)$, ${}_\varepsilon U_n(.;\mathbb{Z}/q)$ and ${}_\varepsilon V_n(.;\mathbb{Z}/q)$ will be respectively denoted $\overline{K}_n(.)$, ${}_\varepsilon\overline{L}_n(.)$, ${}_\varepsilon\overline{U}_n(.)$ and ${}_\varepsilon\overline{V}_n(.)$. Note that for all $n \in \mathbb{Z}$, we have

$$\overline{K}_n(Sf) \simeq \overline{K}_{n-1}(f),\ {}_\varepsilon\overline{L}_n(Sf) \simeq {}_\varepsilon\overline{L}_{n-1}(f),\ {}_\varepsilon\overline{U}_n(Sf) \simeq_\varepsilon \overline{U}_{n-1}(f) \text{ and } {}_\varepsilon\overline{V}_n(Sf) \simeq {}_\varepsilon\overline{V}_{n-1}(f).$$

We also have the following long exact sequences

$$\cdots \longrightarrow \overline{K}_{n+1}(f) \longrightarrow {}_\varepsilon\overline{V}_n(f) \longrightarrow {}_\varepsilon\overline{L}_n(f) \longrightarrow \overline{K}_n(f) \longrightarrow {}_\varepsilon\overline{V}_{n-1}(f) \longrightarrow \cdots$$

$$\cdots \longrightarrow {}_\varepsilon\overline{L}_{n+1}(f) \longrightarrow {}_\varepsilon\overline{U}_n(f) \longrightarrow \overline{K}_n(f) \longrightarrow {}_\varepsilon\overline{L}_n(f) \longrightarrow {}_\varepsilon\overline{U}_{n-1}(f) \longrightarrow \cdots.$$

Proposition 4.1.5. *Let $f : A \longrightarrow B$ be a homomorphism of Hermitian rings. For all $n \in \mathbb{Z}$, we have*

$$_\varepsilon\overline{U}_{n+1}(f) \simeq {}_{-\varepsilon}\overline{V}_n(f).$$

4.2.

Proof. The following diagram of fibrations

$$\begin{array}{ccccc} {}_{-\varepsilon}\mathcal{V}(f) & \longrightarrow & {}_{-\varepsilon}\mathcal{V}(A) & \longrightarrow & {}_{-\varepsilon}\mathcal{V}(B) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_\varepsilon\mathcal{U}(f) & \longrightarrow & \Omega_\varepsilon\mathcal{U}(A) & \longrightarrow & \Omega_\varepsilon\mathcal{U}(B) \end{array}$$

shows that ${}_\varepsilon\overline{U}_{n+1}(f) \simeq {}_{-\varepsilon}\overline{V}_n(f)$ for any $n \geq 2$. For $n < 2$ we have

$$_{-\varepsilon}\overline{V}_n(f) = {}_{-\varepsilon}\overline{V}_2\left(S^{2-n}f\right) \simeq {}_\varepsilon\overline{U}_3\left(S^{2-n}f\right) = {}_\varepsilon\overline{U}_{n+1}(f).$$

Hence, we have proved the proposition. □

Remark 4.2.1. *We define excision in K-Theory with coefficients in $\mathbb{Z}/q$, in a similar way as for the usual K-Theory. As examples of excisive rings for the K-Theory with coefficients in $\mathbb{Z}/q$, we have the following theorem.*

Theorem 4.2.2. [3] *or* [8] *Let I be a ring such that $\widetilde{H}_*(I;\mathbb{Z}/q) = 0$ (I is considered as an abelian group). Then the ring I is excisive for the K-Theory with coefficients in $\mathbb{Z}/q$.*

Remark 4.2.3. *In a similar way, we prove the equivalent of Theorem 3.1.1 for the K-Theory with coefficients in $\mathbb{Z}/q$.*

Corollary 4.2.4. *Let I be a Hermitian ring. If I is excisive for the algebraic K-Theory with coefficients in $\mathbb{Z}/q$, then it is also excisive for the Hermitian K-Theory with coefficients in $\mathbb{Z}/q$.*

4.3.

Proof. Let I be an excisive Hermitian ring for the algebraic K-Theory with coefficients in $\mathbb{Z}/q$ and let the following be a Cartesian diagram of Hermitian rings

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ A_2 & \longrightarrow & A' \end{array}$$

such that $I \simeq \ker \varphi_1$ and φ_1 is surjective. Consider the diagram of exact sequences

$$\begin{array}{ccccccccc} {}_\varepsilon L_2\left(S^2\varphi_2\right) & \xrightarrow{\cdot q} & {}_\varepsilon L_2\left(S^2\varphi_2\right) & \longrightarrow & {}_\varepsilon \overline{L}_2\left(S^2\varphi_2\right) & \longrightarrow & {}_\varepsilon L_1\left(S^2\varphi_2\right) & \xrightarrow{\cdot q} & {}_\varepsilon L_1\left(S^2\varphi_2\right) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}_\varepsilon L_2\left(S^2\varphi_1\right) & \xrightarrow{\cdot q} & {}_\varepsilon L_2\left(S^2\varphi_1\right) & \longrightarrow & {}_\varepsilon \overline{L}_2\left(S^2\varphi_1\right) & \longrightarrow & {}_\varepsilon L_1\left(S^2\varphi_1\right) & \xrightarrow{\cdot q} & {}_\varepsilon L_1\left(S^2\varphi_1\right). \end{array}$$

We have

$${}_\varepsilon L_2\left(S^2\varphi_2\right) \simeq {}_\varepsilon L_0\left(\varphi_2\right) \simeq {}_\varepsilon L_0\left(\varphi_1\right) \simeq {}_\varepsilon L_2\left(S^2\varphi_1\right).$$

We also have

$${}_\varepsilon L_1\left(S^2\varphi_2\right) \simeq {}_\varepsilon L_0\left(S\varphi_2\right) \simeq {}_\varepsilon L_0\left(S\varphi_1\right) \simeq {}_\varepsilon L_1\left(S^2\varphi_1\right).$$

So the diagram shows that

$${}_\varepsilon \overline{L}_0\left(\varphi_2\right) = {}_\varepsilon \overline{L}_2\left(S^2\varphi_2\right) \simeq {}_\varepsilon \overline{L}_2\left(S^2\varphi_1\right) = {}_\varepsilon \overline{L}_0\left(\varphi_1\right).$$

We also prove that

$${}_\varepsilon \overline{L}_{-1}\left(\varphi_2\right) \simeq_\varepsilon \overline{L}_{-1}\left(\varphi_1\right).$$

So according to Remark 4.10 and for all $n \geq -1$, we have

$${}_\varepsilon \overline{L}_n\left(\varphi_2\right) \simeq_\varepsilon \overline{L}_n\left(\varphi_1\right).$$

For $n < -1$, we have

$${}_\varepsilon \overline{L}_n\left(\varphi_2\right) = {}_\varepsilon \overline{L}_0\left(S^{-n}\varphi_2\right) \simeq {}_\varepsilon \overline{L}_0\left(S^{-n}\varphi_1\right) = {}_\varepsilon \overline{L}_n\left(\varphi_1\right).$$

Hence, we have proved the corollary. □

5. Hermitian K-Theory of Stable $\mathbb{C}^*$-algebras

5.1. **Topological K-Theory.**

5.2. Let A be a Banach algebra. The topological K-Theory of A is defined by:

$$K_n^{top}(A) = \begin{cases} \pi_n\left(BGL^{top}(A)\right) & \text{for } n > 0, \\ K_0(A) & \text{for } n = 0. \end{cases}$$

The topology of the space $GL^{top}(A)$ which is the direct limit of $GL_n^{top}(A)$, is induced by the topology of the Banach space $M_n(A)$. For an involutive Banach algebra A, the topological Hermitian K-Theory of A is defined by

$$_{\varepsilon}L_n^{top}(A) = \begin{cases} \pi_n\left(B_{\varepsilon}O^{top}(A)\right) & \text{for } n > 0, \\ _{\varepsilon}L_0(A) & \text{for } n = 0. \end{cases}$$

The canonical map

$$BGL(A) \xrightarrow{\sigma} BGL^{top}(A)$$

induces the map

$$BGL(A)^{+} \xrightarrow{\sigma^{+}} BGL^{top}(A).$$

The following diagram is homotopy commutative

$$\begin{array}{ccc} BGL(A) & \longrightarrow & BGL^{top}(A) \\ \downarrow & \nearrow & \\ BGL(A)^{+}. & & \end{array}$$

For all $n > 0$, by passing to homotopy groups, the map σ^{+} induces the following homomorphisms

$$K_n(A) \xrightarrow{\sigma_n} K_n^{top}(A).$$

For $n = 0$, we let $\sigma_0 = Id$. Similarly, for an involutive Banach algebra A, the canonical map

$$B_{\varepsilon}O(A) \xrightarrow{\tau} B_{\varepsilon}O^{top}(A)$$

induces for all $n > 0$, the following homomorphisms

$$_{\varepsilon}L_n(A) \xrightarrow{\tau_n} {_{\varepsilon}L_n^{top}}(A).$$

For $n = 0$, we let $\tau_0 = Id$. Let A be an involutive Banach algebra. We will denote $\mathcal{K}^{top}(A)$ the space $K_0(A) \times BGL^{top}(A)$ and $_{\varepsilon}\mathcal{L}^{top}(A)$ the space $_{\varepsilon}L_0(A) \times B_{\varepsilon}O^{top}(A)$. Let $_{\varepsilon}\mathcal{U}^{top}(A)$ be the homotopic fiber of the map

$$\mathcal{K}^{top}(A) \longrightarrow {_{\varepsilon}\mathcal{L}^{top}}(A)$$

and let $_{\varepsilon}\mathcal{V}^{top}(A)$ be the homotopic fiber of the map

$$_{\varepsilon}\mathcal{L}^{top}(A) \longrightarrow \mathcal{K}^{top}(A).$$

Then we have the following theorem.

Theorem 5.2.1. [4] *Let A be an involutive Banach algebra. Then it exists a homotopy equivalence between spaces $\Omega_{\varepsilon}\mathcal{U}^{top}(A)$ and $_{-\varepsilon}\mathcal{V}^{top}(A)$.*

For all $n \geq 0$, we let

$${}_{\varepsilon}U_n^{top}(A) = \pi_n\left({}_{\varepsilon}\mathcal{U}^{top}(A)\right) \text{ et } {}_{\varepsilon}V_n^{top}(A) = \pi_n\left({}_{\varepsilon}\mathcal{V}^{top}(A)\right).$$

For all $n \geq 1$, we let

$${}_{\varepsilon}U_n^{top}(A) \simeq {}_{-\varepsilon}V_{n-1}^{top}(A).$$

We also have the diagrams of long exact sequences

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & {}_{\varepsilon}L_{n+1}(A) & \longrightarrow & {}_{\varepsilon}U_n(A) & \longrightarrow & K_n(A) & \longrightarrow & {}_{\varepsilon}L_n(A) & \longrightarrow & {}_{\varepsilon}U_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & {}_{\varepsilon}L_{n+1}^{top}(A) & \longrightarrow & {}_{\varepsilon}U_n^{top}(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & {}_{\varepsilon}L_n^{top}(A) & \longrightarrow & {}_{\varepsilon}U_{n-1}^{top}(A) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & K_{n+1}(A) & \longrightarrow & {}_{\varepsilon}V_n(A) & \longrightarrow & {}_{\varepsilon}L_n(A) & \longrightarrow & K_n(A) & \longrightarrow & {}_{\varepsilon}V_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K_{n+1}^{top}(A) & \longrightarrow & {}_{\varepsilon}V_n^{top}(A) & \longrightarrow & {}_{\varepsilon}L_n^{top}(A) & \longrightarrow & K_n^{top}(A) & \longrightarrow & {}_{\varepsilon}V_{n-1}^{top}(A) & \longrightarrow & \cdots .
\end{array}$$

Definition 5.2.2. *Let A be an involutive Banach algebra. For all $n \geq 0$, we define*

$${}_{\varepsilon}W_n^{top}(A) = \mathit{coker}\left(K_n^{top}(A) \longrightarrow_{\varepsilon} L_n^{top}(A)\right)$$

and

$${}_{\varepsilon}W_n(A) = \mathit{coker}\left(K_n(A) \longrightarrow_{\varepsilon} L_n(A)\right).$$

Proposition 5.2.3. [7] *Let A be an involutive Banach algebra. Then we have*

$${}_{\varepsilon}W_1(A) \simeq {}_{\varepsilon}W_1^{top}(A).$$

Theorem 5.2.4. *Let A be an involutive Banach algebra such that for all $n \geq 0$,*

$$K_n(A) \simeq K_n^{top}(A).$$

Then for all $n \geq 0$ we have

$${}_{\varepsilon}L_n(A) \simeq {}_{\varepsilon}L_n^{top}(A).$$

5.3.

Proof. Let A be an involutive Banach algebra A such that $K_n(A) \simeq K_n^{top}(A)$ for all $n \geq 0$. Consider the following diagram

$$\begin{array}{ccccccc}
K_1(A) & \longrightarrow & {}_{\varepsilon}L_1(A) & \longrightarrow & {}_{\varepsilon}W_1(A) & \longrightarrow & 0 \\
\wr\downarrow & & \downarrow & & \wr\downarrow & & \\
K_1^{top}(A) & \longrightarrow & {}_{\varepsilon}L_1^{top}(A) & \longrightarrow & {}_{\varepsilon}W_1^{top}(A) & \longrightarrow & 0.
\end{array}$$

This diagram proves that

$$\ker\left({}_{\varepsilon}L_1(A) \longrightarrow_{\varepsilon} L_1^{top}(A)\right) \subset \mathrm{Im}\left(K_1(A) \longrightarrow_{\varepsilon} L_1(A)\right)$$

and that the homomorphism ${}_{\varepsilon}L_1(A) \longrightarrow_{\varepsilon} L_1^{top}(A)$ is surjective. The following diagram

$$\begin{array}{ccccccccc}
K_1(A) & \longrightarrow & {}_{\varepsilon}L_1(A) & \longrightarrow & {}_{\varepsilon}U_0(A) & \longrightarrow & K_0(A) & \longrightarrow & {}_{\varepsilon}L_0(A) \\
\wr\downarrow & & \downarrow & & \downarrow & & \| & & \| \\
K_1^{top}(A) & \longrightarrow & {}_{\varepsilon}L_1^{top}(A) & \longrightarrow & {}_{\varepsilon}U_0^{top}(A) & \longrightarrow & K_0^{top}(A) & \longrightarrow & {}_{\varepsilon}L_0^{top}(A)
\end{array}$$

shows that

$${}_{\varepsilon}U_0(A) \simeq {}_{\varepsilon}U_0^{top}(A).$$

Consider the following diagram

$$\begin{array}{ccccccccc}
{}_{\varepsilon}L_1(A) & \longrightarrow & K_1(A) & \longrightarrow & {}_{\varepsilon}V_0(A) & \longrightarrow & {}_{\varepsilon}L_0(A) & \longrightarrow & K_0(A) \\
\downarrow & & \wr\downarrow & & \downarrow & & \| & & \| \\
{}_{\varepsilon}L_1^{top}(A) & \longrightarrow & K_1^{top}(A) & \longrightarrow & {}_{\varepsilon}V_0^{top}(A) & \longrightarrow & {}_{\varepsilon}L_0^{top}(A) & \longrightarrow & K_0^{top}(A).
\end{array}$$

We deduce that for any ε

$${}_{\varepsilon}V_0(A) \simeq {}_{\varepsilon}V_0^{top}(A).$$

Hence for any ε we have

$${}_{\varepsilon}U_1(A) \simeq {}_{\varepsilon}U_1^{top}(A).$$

The following diagram of exact sequences

$$\begin{array}{ccccccccc}
{}_{\varepsilon}U_1(A) & \longrightarrow & K_1(A) & \longrightarrow & {}_{\varepsilon}L_1(A) & \longrightarrow & {}_{\varepsilon}U_0(A) & \longrightarrow & K_0(A) \\
\wr\downarrow & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \| \\
{}_{\varepsilon}U_1^{top}(A) & \longrightarrow & K_1^{top}(A) & \longrightarrow & {}_{\varepsilon}L_1^{top}(A) & \longrightarrow & {}_{\varepsilon}U_0^{top}(A) & \longrightarrow & K_0^{top}(A)
\end{array}$$

proves that

$${}_{\varepsilon}L_1(A) \simeq {}_{\varepsilon}L_1^{top}(A).$$

Then we prove the result, proceeding as in Section 3.5. □

Definition 5.3.1. *Let $\mathcal{K}$ be the $\mathbb{C}^*$-algebra of the compact operators on the standard separable Hilbert space. We say that a $\mathbb{C}^*$-algebra A is stable if and only if it is isomorphic to $\mathcal{K} \otimes A$.*

Theorem 5.3.2. [12] *Let A be a stable $\mathbb{C}^*$-algebra. The homomorphism*

$$K_n(A) \xrightarrow{\sigma_n} K_n^{top}(A)$$

is an isomorphism for all $n \geq 0$.

The following theorem is a direct consequence of the two preceding theorems.

Theorem 5.3.3. *Let A be an involutive stable $\mathbb{C}^*$-algebra. For all $n \geq 0$, we have*

$$_{\varepsilon}L_n(A) \simeq {}_{\varepsilon}L_n^{top}(A).$$

Example 5.3.4. *Let $\mathcal{K}$ be the $\mathbb{C}^*$-algebra of the compact operators on the standard separable Hilbert space $\mathcal{H}$. Let $A = C(X; \mathcal{K})$ be the $\mathbb{C}^*$-algebra of the continuous functions from a compact space X to $\mathcal{K}$. This algebra is stable. The density theorem* [5] *proves that*

$$_1L_n^{top}(A) \simeq {}_1L_n^{top}(C(X; \mathbb{C})).$$

Definition 5.3.5. *Let Λ be an involutive Banach algebra. We say that Λ is a C-algebra if, for every $x \in M_n(\Lambda)$, $1 + x\overline{x} \in GL_n(\Lambda)$.*

In [5, *p.* 234], Karoubi proves that for a C -algebra B, there is a natural isomorphism

$$_1L_n^{top}(B) \simeq K_n^{top}(B) \oplus K_n^{top}(B).$$

Proposition 5.3.6. *Since $C(X; \mathbb{C})$ is a C-algebra, for all $n \geq 0$, we have the following isomorphism*

$$_1L_n(A) \simeq K_n(X) \oplus K_n(X).$$

Example 5.3.7. *If X is the complex projective space $\mathbb{C}P^r$, we obtain*

$$_1L_n(A) = \begin{cases} \mathbb{Z}^{2r} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

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