GENERALIZED-DESIGNS

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Abstract. The generalization of designs can be considered as infinite designs. A finite set can be a part of an infinite set, a well-ordered set, a partial ordered set, or even a (discrete) topological space; in this paper designs are generalized due to this point of view. In this way for a mathematical structure $\mathcal{M}$, $\mathcal{M}-$designs are developed, especially some results in poset-designs are obtained.

1. Introduction

It is well-known that a finite $(\upsilon, \kappa, \lambda)$ design is a collection $\boldsymbol{A}$ of subsets of a set S such that $\kappa < \upsilon = \text{card}(S)$ $(\upsilon, \kappa, \lambda \in \mathbb{N})$, each $B \in \boldsymbol{A}$ has κ elements, and for each $x \neq y$ there are exactly λ members of $\boldsymbol{A}$ containing x and y [1]. The most obvious way to generalize a finite design is to substitute finite sets by arbitrary sets. Since all of the well-ordered structures on a finite set have the same ordinal number which is the same as its cardinal number, the definition of a (finite) design can be generalized by considering well-ordered sets. In this way it is clear that if two subsets of a finite set are equipotent, then their complements are equipotent too; but if two subsets of a well-ordered set have the same ordinal number, their complements may have the same ordinal number or not. So in considering a subset of a well-ordered set one may be interested in the ordinal number of the complement of that subset too; this point of view causes four types of a (well-ordered)-design. In addition, a well-ordered set is a poset (*partial ordered set*) so one may consider poset-designs too, moreover a finite set can be regarded as a discrete topological space, consequently a TOP-design can be defined. Now this question arises: If $\mathcal{M}$ is a mathematical structure, what should be the definition of $\mathcal{M}-$designs?

In this paper first a (finite) design is generalized in cardinal point of view in which the reader may find similar ideas but with different proofs to [3]. In other sections the concept of (well-ordered)-designs is introduced, then other generalizations like TOP-designs are developed and Section 5 will be

Dedicated to Dr N. Izaddoustdar, with best wishes for him.

ended with some nice examples. The last section contains the conclusion. In this paper AC+CH is assumed.

2. Preliminaries

For any set D, a unique cardinal number denoted by $\mathrm{card}(D)$ which is in one-to-one correspondence with D, is assigned. Let α, β be two cardinal numbers $\alpha + \beta$ is the cardinal number of the set. $(\alpha \times \{0\}) \cup (\beta \times \{1\})$, $\alpha\beta$ is the cardinal number of $\alpha \times \beta$, and α^β is the cardinal number of the set $\{f | f : \beta \to \alpha \text{ is a function}\}$. Moreover if α, β are two cardinal numbers we use $\alpha \leq \beta$ if there exists an injection $f : \alpha \to \beta$. For any two cardinal numbers α, β, exactly one of the relations $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ holds ($\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$). For two nonzero cardinal numbers α, β such that at least one of them is transfinite, we have $\alpha\beta = \alpha + \beta = \max(\alpha, \beta)$. In addition the set of all subsets of D has cardinal number $2^{\mathrm{card}(D)}$ and $\mathrm{card}(D) < 2^{\mathrm{card}(D)}$.

For any well-ordered set $(D, \leq)$, a unique ordinal number denoted by $\mathrm{ord}(D, \leq)$ (or briefly $\mathrm{ord}(D)$) such that there exists an order preserving bijection between $(D, \leq)$ and $\mathrm{ord}(D, \leq)$ is assigned. Let α, β be two ordinal numbers. $\alpha+\beta$ is the ordinal number of $((\alpha \times \{0\}) \cup (\beta \times \{1\}), \preceq)$ where for all $(x, y), (z, w) \in (\alpha \times \{0\}) \cup (\beta \times \{1\})$ we have $(x, y) \preceq (z, w)$ if and only if $(y = 0 \wedge w = 1) \vee (y = w \wedge x \leq z)$ (note that if $y = w$, then $x, z \in \alpha$ or $x, z \in \beta$), $\alpha\beta$ is the ordinal number of $(\beta \times \alpha, \preceq)$ where for all $(x, y), (z, w) \in \beta \times \alpha$ we have $(x, y) \preceq (z, w)$ if and only if $x < z \vee (x = z \wedge y \leq w)$. It is well-known that for any ordinal numbers α, β, γ we have $\alpha + \beta = \alpha + \gamma$ if and only if $\beta = \gamma$ and for $\alpha \neq 0$, $\alpha\beta = \alpha\gamma$ if and only if $\beta = \gamma$. Moreover if α, β are two ordinal numbers we use $\alpha \leq \beta$ if there exists an order-preserving injection $f : \alpha \to \beta$. For any two ordinal numbers $\alpha \leq \beta$ exactly one of the relations $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$ holds. Any nonempty collection of ordinal (resp. cardinal) numbers has a minimum, therefore any collection of ordinal (resp. cardinal) numbers with an upper bound has supremum.

We refer the interested reader on cardinal and ordinal arithmetic to [7].

3. Cardinal Numbers' Point of View

Let $\mathrm{card}(\mathbb{N}) = \aleph_0$, $\mathrm{card}(\mathbb{R}) = c$, moreover for a transfinite cardinal number ξ by $\xi - \alpha$ we mean ξ (where $\xi > \alpha$).

In this section some results similar to [3] are obtained.

Definition 3.1. By a $(\upsilon, \kappa, \lambda)$ design we mean a collection $\boldsymbol{A}$ of subsets of a set S such that $\kappa \leq \upsilon = \mathrm{card}(S)$ (υ, κ, and λ are nonzero cardinal numbers) and:

- $\forall B \in \boldsymbol{A} \quad \mathrm{card}(B) = \kappa$;
- $\forall x \in S \quad \forall y \in S - \{x\} \quad \mathrm{card}(\{B \in \boldsymbol{A} | x, y \in B\}) = \lambda$.

Restriction: In this section we assume $1 < \kappa < \upsilon = \text{card}(S)$.

Theorem 3.2. Let $\boldsymbol{A}$ be a $(\upsilon, \kappa, \lambda)$ design on S and $\text{card}(\boldsymbol{A}) = b$. We have [3]:

1. $\exists r \quad \forall x \in S \quad \text{card}(\{B \in \boldsymbol{A} | x \in B\}) = r$;

For the other item, suppose r is such that:

$$\forall x \in S \quad \text{card}(\{B \in \boldsymbol{A} | x \in B\}) = r;$$

2. $(\upsilon - 1)\lambda = r(\kappa - 1)$ and if υ is a transfinite cardinal number, then $\upsilon\lambda = r\kappa = r$.

Proof. Let $x \in S$. Let $t \in S - \{x\}$ and $H \in \boldsymbol{A}$ be such that $x, t \in H$. For each $z \in S - \{x\}$ choose a bijection $\Psi_x : \{B \in \boldsymbol{A} | x, t \in B\} \to \{B \in \boldsymbol{A} | z, x \in B\}$; for each $K \in \boldsymbol{A}$ such that $x \in K$ choose a bijection $\Phi_K : K - \{x\} \to H - \{x\}$. Define injection $\Lambda : \{B \in \boldsymbol{A} | x, t \in B\} \times (S - \{x\}) \to \{B \in \boldsymbol{A} | x \in B\} \times (H - \{x\})$ such that for each $(T, z) \in \{B \in \boldsymbol{A} | x, t \in B\} \times (S - \{x\})$, $\Lambda(T, z) = (\Psi_z(T), \Phi_{\Psi_z(T)}(z))$; so for each $(K, w) \in \{B \in \boldsymbol{A} | x \in B\} \times (H - \{x\})$, there exists $s \in K - \{x\}$ and $L \in \{B \in \boldsymbol{A} | x, t \in B\}$ such that $\Phi_K(s) = w$ and $\Psi_s(L) = K$, so $\Lambda(L, s) = (K, w)$ and $\Lambda : \{B \in \boldsymbol{A} | x, t \in B\} \times (S - \{x\}) \to \{B \in \boldsymbol{A} | x \in B\} \times (H - \{x\})$ is a bijection, thus $(\upsilon - 1)\lambda = \text{card}(\{B \in \boldsymbol{A} | x \in B\})(\kappa - 1)$ and by a similar method $(\upsilon - 1)\lambda = \text{card}(\{B \in \boldsymbol{A} | t \in B\})(\kappa - 1)$, we have the following cases:

- υ is a finite cardinal number. In this case κ is also a finite cardinal number and using $\text{card}(\{B \in \boldsymbol{A} | t \in B\})(\kappa - 1) = \text{card}(\{B \in \boldsymbol{A} | x \in B\})(\kappa - 1)$, we have:
$$\text{card}(\{B \in \boldsymbol{A} | t \in B\}) = \text{card}(\{B \in \boldsymbol{A} | x \in B\}).$$
- υ is a transfinite cardinal number. In this case we have $\upsilon\lambda = \text{card}(\{B \in \boldsymbol{A} | t \in B\})\kappa$, so κ is a transfinite cardinal number or $\text{card}(\{B \in \boldsymbol{A} | t \in B\})$ is a transfinite cardinal number and $\max(\upsilon, \lambda) = \upsilon\lambda = \text{card}(\{B \in \boldsymbol{A} | t \in B\})\kappa = \max(\text{card}(\{B \in \boldsymbol{A} | t \in B\}), \kappa)$, using $\kappa < \upsilon$, we have $\kappa < \max(\upsilon, \lambda) = \upsilon\lambda = \text{card}(\{B \in \boldsymbol{A} | t \in B\})$, by a similar method $\upsilon\lambda = \text{card}(\{B \in \boldsymbol{A} | x \in B\})$ and:

$$\kappa < \max(\upsilon, \lambda) = \upsilon\lambda = \text{card}(\{B \in \boldsymbol{A} | t \in B\})\text{card}(\{B \in \boldsymbol{A} | x \in B\}).$$

So we can choose r such that:

$$\forall x \in S \quad \text{card}(\{B \in \boldsymbol{A} | x \in B\}) = r.$$

□

Definition 3.3. By a $t-(\upsilon, \kappa, \lambda)$ design we mean a collection $\boldsymbol{A}$ of subsets of a set S such that $t < \kappa < \upsilon = \text{card}(S)$ (υ, κ, λ, and t are nonzero cardinal numbers) and:

- $\forall B \in \boldsymbol{A} \quad \text{card}(B) = \kappa$;

- $\forall C \subseteq S \quad (0 < \mathrm{card}(C) = t \Rightarrow \mathrm{card}(\{B \in \boldsymbol{A} | C \subseteq B\}) = \lambda)$.

Theorem 3.2 can be regarded as a special case of this theorem, but since Theorem 3.2 is much more similar to famous theorems in (finite) design theory, it's better to separate Theorem 3.2 and the following Theorem (moreover in Theorem 3.2 more specific relations have been obtained).

Theorem 3.4. Let $\boldsymbol{A}$ be a $t-(\upsilon,\kappa,\lambda)$ design on S and $\mathrm{card}(A) = b$ [3].

(1) if λ is finite, then t is finite;
(2) if t is transfinite, then λ is transfinite and $\lambda \geq \upsilon$;
(3) if "t is finite or $\upsilon^\kappa = \upsilon$" and $0 < \mu < t$, then there exists r_μ such that $\boldsymbol{A}$ is a $\mu-(\upsilon,\kappa,r_\mu)$ design on S.

Proof. Clearly if υ is transfinite, then $\lambda \leq b \leq \upsilon^\kappa \leq 2^\upsilon$ on the other hand $\bigcup \boldsymbol{A} = S$ thus $\upsilon = \mathrm{card}(\bigcup \boldsymbol{A}) \leq b\kappa \leq \upsilon^\kappa \upsilon = \upsilon^\kappa$, so b is transfinite and $\max(\lambda, \upsilon) \leq b \leq \upsilon^\kappa$.

1. Suppose λ is finite and t is transfinite, choose $X \subseteq S$ such that $\mathrm{card}(X) = t$, there exist $B_1, ..., B_\lambda \in \boldsymbol{A}$ such that $\{B_1, ..., B_\lambda\} = \{B \in \boldsymbol{A} | X \subseteq B\}$, so for each $x \in S$ we have $\{B_1, ..., B_\lambda\} = \{B \in \boldsymbol{A} | X \cup \{x\} \subseteq B\}$ (since $\mathrm{card}(X \cup \{x\}) = t$ and $X \subseteq X \cup \{x\}$), thus $\{B_1, ..., B_\lambda\} = \{S\}$, which is a contradiction since $\kappa = \mathrm{card}(B_1) < \mathrm{card}(S) = \upsilon$.
2. Suppose t is transfinite $C \subseteq S$, $\mathrm{card}(C) = t$ and $\lambda < \upsilon$, so $\mathrm{card}(\bigcup\{B \in \boldsymbol{A} | C \subseteq B\}) \leq \lambda\kappa < \upsilon$, therefore there exists $D \subseteq S$ such that $\mathrm{card}(D) = t$ and $D \cap (\bigcup\{B \in \boldsymbol{A} | C \subseteq B\}) = \emptyset$, since $\mathrm{card}(C \cup D) = t$ and $0 < \lambda$, there exists $B \in \boldsymbol{A}$ such that $C \cup D \subseteq B$, which is a contradiction, thus $\lambda \geq \upsilon$.
3. We have the following cases
   - t is transfinite and $\upsilon^\kappa = \upsilon$. By (2) and $\max(\lambda, \upsilon) \leq b \leq \upsilon^\kappa$, $\lambda = \upsilon = b$; since each μ subset of S can be extended to a t subset of S, we have the desired result by $r_\mu = \upsilon$.
   - t is finite. Suppose $x_1, ..., x_{t-1}, x_t$ be t distinct elements of S, choose $H \in \boldsymbol{A}$ such that $x_1, ..., x_{t-1}, x_t \in H$. For each $z \in S - \{x_1, ..., x_{t-1}, x_t\}$ there exists a bijection $\Psi_z : \{B \in \boldsymbol{A} | x_1, ..., x_{t-1}, z \in B\} \to \{B \in \boldsymbol{A} | x_1, ..., x_{t-1}, x_t \in B\}$ and for each $K \in \boldsymbol{A}$ such that $x_1, ..., x_{t-1} \in K$ there exists a bijection $\Phi_K : K - \{x_1, ..., x_{t-1}\} \to H - \{x_1, ..., x_{t-1}\}$, define $\Lambda : (S - \{x_1, ..., x_{t-1}\}) \times \{B \in \boldsymbol{A} | x_1, ..., x_{t-1}, x_t \in B\} \to \{B \in \boldsymbol{A} | x_1, ..., x_{t-1} \in B\} \times (H - \{x_1, ..., x_{t-1}\})$ with $\Lambda(z, T) = (\Psi_z(T), \Phi_{\Psi_z(T)}(z))$ $(\forall (z, T) \in (S - \{x_1, ..., x_{t-1}\}) \times \{B \in \boldsymbol{A} | x_1, ..., x_{t-1}, x_t \in B\})$. Using a similar method described in Theorem 3.2 will complete the proof, moreover $(\upsilon - (t-1))\lambda = r_{t-1}(\kappa - (t-1))$, so if υ is transfinite, then $\upsilon\lambda = r_{t-1}(\kappa - (t-1)) = r_{t-1}$, $\kappa < \upsilon \leq r_{t-1}$, and $\boldsymbol{A}$ is a $(t-1)-(\upsilon,\kappa,r_{t-1})$ design on S.

□

Definition 3.5. With the same assumptions as in Theorem 3.2:

- $\Gamma : \boldsymbol{A} \times S \to \{0,1\}$ such that $\Gamma(B,x) = 1$ if and only if $x \in B$ ($x \in S, B \in \boldsymbol{A}$), is called the incidence matrix of $(\upsilon, \kappa, \lambda)$ design $\boldsymbol{A}$.
- $\boldsymbol{A}$ is called resolvable if there is a partition Λ of $\boldsymbol{A}$ such that $\mathrm{card}(\Lambda) = r$, for each $\Omega, \Theta \in \Lambda$ we have $\mathrm{card}(\Omega) = \mathrm{card}(\Theta)$, and for each $x \in S$ there exists a unique $B \in \Omega$ such that $x \in B$ (i.e., Ω is a partition of S).

Corollary 3.6. With the same assumptions as in Theorem 3.2, if $\Gamma : \boldsymbol{A} \times S \to \{0,1\}$ is the incidence matrix of $\boldsymbol{A}$, and $\Gamma' : S \times \boldsymbol{A} \to \{0,1\}$ is $\Gamma'(x,B) = \Gamma(B,x)$ ($x \in S, B \in \boldsymbol{A}$), then for each $x, y \in S$ we have:

$$\mathrm{card}(\{B \in \boldsymbol{A} | \Gamma(B,x)\Gamma'(y,B) \neq 0\}) = \begin{cases} r & x = y, \\ \lambda & x \neq y. \end{cases} .$$

Example 3.7. $\boldsymbol{A} = \{\{x, x+y\} | x, y \in \mathbb{N}\}$ is an $(\aleph_0, 2, 1)$ resolvable design on $\mathbb{N}$. Consider $\boldsymbol{A}$ as the well-ordered set $(\boldsymbol{A}, \leq_0)$ such that for each $n, m, k, l \in \mathbb{N}$, $\{n, n+m\} \leq_0 \{k, k+l\}$ if and only if "$n < k \vee (n = k \wedge m < l)$" (i.e., $\boldsymbol{A} = \{\{1,2\}, \{1,3\}, ..., \{2,3\}, \{2,4\}, ..., \{n, n+1\}, \{n, n+2\}, ...\}$), define $\vartheta_1 : \mathbb{N} \to \boldsymbol{A}$ such that:

- $\vartheta_1(1) = \{1, 2\}$,
- $\vartheta_1(n) = \min\limits_{\leq_0}\{\{x,y\} \in \boldsymbol{A} | \forall k \in \{1, ..., n-1\}\ \{x,y\} \cap \vartheta_1(k) = \emptyset\}$ $(n > 1)$,

(so $\vartheta_1(n) = \{2n-1, 2n\}$ $(n \in \mathbb{N})$); now for each $m > 1$, define $\vartheta_m : \mathbb{N} \to \boldsymbol{A}$ such that:

- $\vartheta_m(1) = \{1, m+1\}$,
- $\vartheta_m(n) = \min\limits_{\leq_0}\{\{x,y\} \in \boldsymbol{A} | (\forall k \in \{1, ..., n-1\}\ \{x,y\} \cap \vartheta_m(k) = \emptyset) \wedge (\forall k \in \{1, ..., m-1\}\ \forall l \in \mathbb{N}\ \{x,y\} \neq \vartheta_k(l))\}$ $(n > 1)$,

$\{\vartheta_m(\mathbb{N}) : m \in \mathbb{N}\}$ is a desired partition of $\boldsymbol{A}$.

Example 3.8. $\boldsymbol{A} = \{\{x, y, z, 1\} | 1 < x < y < z\}$ is $1 - (\aleph_0, 4, \aleph_0)$ and $2 - (\aleph_0, 4, \aleph_0)$ design on $\mathbb{N}$ (of course not a nice one, since $r = \mathrm{card}(\boldsymbol{A}) = \aleph_0$) [3, Example 2], and it is not resolvable.

Example 3.9. Suppose $\boldsymbol{A}$ contains the following sets:

$$\begin{array}{ccccccc} \{1,2,3\} & \{1,4,5\} & \{1,6,7\} & \{1,8,9\} & \{1,10,11\} & \{1,12,13\} & \cdots \\ \{2,4,6\} & \{2,5,7\} & \{2,8,10\} & \{2,9,11\} & \{2,12,14\} & \{2,13,15\} & \cdots \\ \{3,4,7\} & \{3,5,6\} & \{3,7,8\} & \{3,9,10\} & \{3,11,12\} & \{3,13,14\} & \cdots \\ \{4,8,11\} & \{4,9,12\} & \{4,10,13\} & \{4,14,16\} & \{4,15,17\} & \cdots & \\ \vdots & & & & & & \end{array}$$

which is produced by induction, then $\boldsymbol{A}$ is a $2 - (\aleph_0, 3, 1)$ design on $\mathbb{N}$.

The interested reader can refer to [2], [4], and [5] for more related examples and discussions.

4. Partial Ordered Set (Poset) and Ordinal Numbers' Point of View

In a partial ordered set $(S, \leq)$ for $U \subseteq S$, $\leq_U$ denotes the partial order on U induced by $\leq$, moreover if $(S, \leq)$ is a well-ordered set, then $\mathrm{ord}(U, \leq_U) \leq \mathrm{ord}(S, \leq)$ and $\alpha = \mathrm{ord}(S, \leq)$ is called an initial ordinal number if $\alpha = \min\{\mathrm{ord}(S, \leq_0) | (S, \leq_0) \text{ is a well-ordered set}\}$. For partial ordered sets $(O_1, \leq_1)$ and $(O_2, \leq_2)$ we write $(O_1, \leq_1) \approx (O_2, \leq_2)$ (or $O_1 \approx O_2$) if there exists an order-preserving bijective map $f : (O_1, \leq_1) \to (O_2, \leq_2)$ such that $f^{-1} : (O_2, \leq_2) \to (O_1, \leq_1)$ is order preserving too. Suppose $\mathrm{ord}(\mathbb{N}, \leq) = \omega$.

Definition 4.1. Consider (well-ordered)-designs in the following points of view:

- **1st type:** By a $t-(\upsilon, (\kappa_1, \kappa_2), \lambda)$ (well-ordered)-design, we mean a collection $\boldsymbol{A}$ of subsets of well-ordered set $S = \{\alpha | \alpha < \upsilon\}$ such that $\max(t, \kappa_1, \kappa_2) \leq \upsilon$ (t, υ, κ_1, and κ_2 are nonzero ordinal numbers, λ is a nonzero cardinal number) and:
  - $\forall B \in \boldsymbol{A} \quad (\mathrm{ord}(B) = \kappa_1 \wedge \mathrm{ord}(S - B) = \kappa_2)$;
  - $\forall C \subseteq S \quad (\mathrm{ord}(C) = t \Rightarrow \mathrm{card}(\{B \in \boldsymbol{A} | C \subseteq B\}) = \lambda)$.
- **2nd type:** By a $t-(\upsilon, \kappa, \lambda)$ (well-ordered)-design, we mean a collection $\boldsymbol{A}$ of subsets of a well-ordered set $S = \{\alpha | \alpha < \upsilon\}$ such that $\max(\kappa, t) \leq \upsilon$ (t, υ, and κ are nonzero ordinal numbers, λ is a nonzero cardinal number) and:
  - $\forall B \in \boldsymbol{A} \quad \mathrm{ord}(B) = \kappa$;
  - $\forall C \subseteq S \quad (\mathrm{ord}(C) = t \Rightarrow \mathrm{card}(\{B \in \boldsymbol{A} | C \subseteq B\}) = \lambda)$.
- **3rd type:** By a $(t_1, t_2)-(\upsilon, (\kappa_1, \kappa_2), \lambda)$ (well-ordered)-design, we mean a collection $\boldsymbol{A}$ of subsets of a well-ordered set $S = \{\alpha | \alpha < \upsilon\}$ such that $\max(t_1, t_2, \kappa_1, \kappa_2) \leq \upsilon$ (t_1, t_2, υ, κ_1), and κ_2 are nonzero ordinal numbers, λ is a nonzero cardinal number) and:
  - $\forall B \in \boldsymbol{A} \quad (\mathrm{ord}(B) = \kappa_1 \wedge \mathrm{ord}(S - B) = \kappa_2)$;
  - $\forall C \subseteq S \quad ((\mathrm{ord}(C) = t_1 \wedge \mathrm{ord}(S - C) = t_2) \Rightarrow \mathrm{card}(\{B \in \boldsymbol{A} | C \subseteq B\}) = \lambda)$.
- **4th type:** By a $(t_1, t_2)-(\upsilon, \kappa, \lambda)$ (well-ordered)-design, we mean a collection $\boldsymbol{A}$ of subsets of well-ordered set $S = \{\alpha | \alpha < \upsilon\}$ such that $\max(\kappa, t_1, t_2) \leq \upsilon$ (t_1, t_2, υ), and κ are nonzero ordinal numbers, λ is a nonzero cardinal number and:
  - $\forall B \in \boldsymbol{A} \quad \mathrm{ord}(B) = \kappa$;
  - $\forall C \subseteq S \quad ((\mathrm{ord}(C) = t_1 \wedge \mathrm{ord}(S - C) = t_2) \Rightarrow \mathrm{card}(\{B \in \boldsymbol{A} | C \subseteq B\}) = \lambda)$.

Comparison 4.2. Now we will compare the above definitions. Set the following predications:

- θ_1: $\boldsymbol{A}$ is a $t_1 - (\upsilon, (\kappa_1, \kappa_2), \lambda)$ (well-ordered)-design (1st type).
- θ_2: $\boldsymbol{A}$ is a $t_1 - (\upsilon, \kappa_1, \lambda)$ (well-ordered)-design (2nd type).
- θ_3: $\boldsymbol{A}$ is a $(t_1, t_2) - (\upsilon, (\kappa_1, \kappa_2), \lambda)$ (well-ordered)-design (3rd type).
- θ_4: $\boldsymbol{A}$ is a $(t_1, t_2) - (\upsilon, \kappa_1, \lambda)$ (well-ordered)-design (4th type).

And,

- π_1: If t_1 is finite, then $\boldsymbol{A}$ is a $t_1 - (\upsilon', \kappa', \lambda)$ design, where $\upsilon' = \text{card}(\{\alpha|\alpha < \upsilon\})$ and $\kappa' = \text{card}(\{\alpha|\alpha < \kappa_1\})$ (similarity to the cardinal point of view).
- π_2: Let $\bigcup \boldsymbol{A} = \{\alpha|\alpha < \upsilon\}$, t_1 is finite and υ is not finite. If κ_1 is a limit ordinal number, then υ is a limit ordinal number.
- π_3: Let α, β be ordinal numbers such that $\upsilon = \alpha\kappa_1 + \beta$ and $0 \leq \beta < \kappa_1$, then $\beta < t_1$ and $\boldsymbol{A}$ is a $t_1 - (\upsilon - \beta, (\kappa_1, \kappa_2 - \beta), \lambda)$ (well-ordered)-design. So for each $t_1 < \tau < \kappa_1$ there isn't any $t_1 - (\alpha\kappa_1 + \tau, (\kappa_1, \kappa_2), \lambda)$ (well-ordered)-design.
- π_4: For initial infinite ordinal number υ we have: $\bigcup \boldsymbol{A} = \{\alpha|\alpha < \upsilon\}$.
- π_5: For initial infinite ordinal number υ we have: If $\kappa_1 < \upsilon$, then $\kappa_2 = \upsilon$.

The following table is valid:

j \ i	1	2	3	4
1	$\surd$	$\surd$	$\surd$	$\surd$
2	$\surd$	$\surd$	$\surd$	$\surd$
3	$\surd$			
4	$\surd$	$\surd$	$\surd$	$\surd$
5	$\surd$		$\surd$	

Table 1

In the above table the mark "$\surd$" indicates that in the corresponding case we have: "$\theta_i \Rightarrow \pi_j$".

Example 4.3. Let Ω be the least uncountable ordinal number and $\boldsymbol{A} = \{B \subseteq \{\alpha|\alpha \leq \Omega\}|\text{ord}(B) = \omega\omega\}$, then $\boldsymbol{A}$ is an $\omega - (\Omega + 1, (\omega\omega, \Omega + 1), c)$ (well-ordered)-design (in the 1st point of view) and it is not a $1 - (\Omega + 1, (\omega\omega, \Omega + 1), c)$ (well-ordered)-design moreover it is not an $\aleph_0 - (c, \aleph_0, c)$ design.

Definition 4.4. Let O be a partial ordered set, $C, D \subseteq O$ and consider poset-designs in the following points of view:

- **1st type:** By a $C - (O, D, \lambda)$ poset-design, we mean a collection $\boldsymbol{A}$ of subsets of poset O such that (λ is a nonzero cardinal number):

  - $\forall B \in \boldsymbol{A} \quad (B \approx D \wedge O - B \approx O - D)$;
  - $\forall E \subseteq O \quad (E \approx C \Rightarrow \text{card}(\{B \in \boldsymbol{A} | E \subseteq B\}) = \lambda)$. $(*)$
- **2nd type:** By a $C - (O, D, \lambda)$ poset-design, we mean a collection $\boldsymbol{A}$ of subsets of poset O such that (λ is a nonzero cardinal number):
  - $\forall B \in \boldsymbol{A} \quad B \approx D$;
  - $\forall E \subseteq O \quad (E \approx C \Rightarrow \text{card}(\{B \in \boldsymbol{A} | E \subseteq B\}) = \lambda)$.
- **3rd type:** By a $C - (O, D, \lambda)$ poset-design, we mean a collection $\boldsymbol{A}$ of subsets of poset O such that (λ is a nonzero cardinal number):
  - $\forall B \in \boldsymbol{A} \quad (B \approx D \wedge O - B \approx O - D)$;
  - $\forall E \subseteq O \quad ((E \approx C \wedge O - E \approx O - C) \Rightarrow \text{card}(\{B \in \boldsymbol{A} | E \subseteq B\}) = \lambda)$.
- **4th type:** By a $C - (O, D, \lambda)$ poset-design, we mean a collection $\boldsymbol{A}$ of subsets of poset O such that (λ is a nonzero cardinal number):
  - $\forall B \in \boldsymbol{A} \quad B \approx D$;
  - $\forall E \subseteq O \quad ((E \approx C \wedge O - E \approx O - C) \Rightarrow \text{card}(\{B \in \boldsymbol{A} | E \subseteq B\}) = \lambda)$.

Remark 4.5.

- If $\leq = \{(x, x) | x \in O\}$, then in the partial ordered set $(O, \leq)$, for $C, D \subseteq O$, we have:

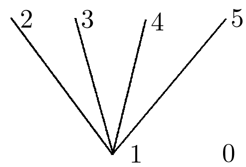
  $\boldsymbol{A}$ is a $C - (O, D, \lambda)$ poset -design (in all of the above points of view)
  if and only if
  $\boldsymbol{A}$ is a a $\text{card}(C) - (\text{card}(O), \text{card}(D), \lambda)$ design.

- If $(O, \leq)$ is a well-ordered set, then:
  - $\boldsymbol{A}$ is a $C - (O, D, \lambda)$ poset-design of the 1st type if and only if $\boldsymbol{A}$ is an $\text{ord}(C) - (\text{ord}(O), (\text{ord}(D), \text{ord}(O - D)), \lambda)$ (well-ordered)-design of the 1st type.
  - $\boldsymbol{A}$ is a $C - (O, D, \lambda)$ poset-design of the 2nd type if and only if $\boldsymbol{A}$ is an $\text{ord}(C) - (\text{ord}(O), \text{ord}(D), \lambda)$ (well-ordered)-design of the 2nd type.
  - $\boldsymbol{A}$ is a $C - (O, D, \lambda)$ poset-design of the 3rd type if and only if $\boldsymbol{A}$ is an $(\text{ord}(C), \text{ord}(O - C)) - (\text{ord}(O), (\text{ord}(D), \text{ord}(O - D))\lambda)$ (well-ordered)-design of the 3rd type.
  - $\boldsymbol{A}$ is a $C - (O, D, \lambda)$ poset-design of the 4th type if and only if $\boldsymbol{A}$ is an $(\text{ord}(C), \text{ord}(O - C)) - (\text{ord}(O), \text{ord}(D), \lambda)$ (well-ordered)-design of the 4th type.
- Let ρ_i denote the prediction $\boldsymbol{A}$ is a $C - (O, D, \lambda)$ poset -design of the ith. type, then we have the following diagram (thus also for

(well-ordered)-designs):

$$\begin{array}{ccc} \rho_1 & \Rightarrow & \rho_3 \\ \Downarrow & & \Downarrow \\ \rho_2 & \Rightarrow & \rho_4 \end{array} .$$

Counterexample 4.6. Consider $O = \{0, 1, 2, 3, 4, 5\}$, ordered as the following diagram (i.e., $\leq = \{(x, x)|x \in O\} \cup \{(1, x)|x \in \{2, 3, 4, 5\}\}$):



for $C = D = \{2, 3\}$, we have:

- $\boldsymbol{A} = \{\{x, y\}|x \not\leq y, y \not\leq x\}$, is a $C - (O, D, 1)$ poset-design of the 2nd type, but not of the 1st and 3rd type.
- $\boldsymbol{A} = \{\{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$, is a $C - (O, D, 1)$ poset-design of the 3rd type, but not of the 1st and 2nd type.
- $\boldsymbol{A} = \{\{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{2, 0\}\}$, is a $C-(O, D, 1)$ poset-design of the 4th Type, but not of the 1st, 2nd, and 3rd type.

Thus the reverse of implications in Remark 4.5 is not true.

Remark 4.7. In our way we will be able to define semi-upper poset-designs, e.g., for the 1st type the line $(*)$ in Definition 4.4, may be replaced by: "$\forall E \subseteq O \quad (E \approx C \Rightarrow \text{card}(\{B \in \boldsymbol{A}|E \subseteq B\}) \geq \lambda)$"; by a same method semi-lower poset-designs.

5. **Generalizations**

Generalization 5.1.

(1) Definitions 4.1 and 4.4, suggest the above definitions be generalized for other structures, e.g., Topological Spaces, i.e., in the 1st point of view if X is a topological space and C, D are its subspaces, then by a $C - (X, D, \lambda)$ TOP-design (1st type), we mean a collection $\boldsymbol{A}$ of subsets of X such that (λ is a nonzero cardinal number):
- $\forall B \in \boldsymbol{A} \quad (B \approx D \wedge X - B \approx X - D)$,
- $\forall E \subseteq X \quad (E \approx C \Rightarrow \text{card}(\{B \in \boldsymbol{A}|E \subseteq B\}) = \lambda)$,

where for topological spaces S and T, by $S \approx T$, we mean S and T are homeomorph; for other types use similar definitions.

(2) It is clear that a $t - (\upsilon, \kappa, \lambda)$ design, defined in Definition 3.3, indicates a generalized set-design of the 2nd type.

Counterexample 5.2. For $X = [0, 1]$ with the induced topology of $\mathbb{R}$ and $C = \left\{0, \frac{1}{3}\right\}$, $D = \left\{0, \frac{1}{2}, 1\right\}$, we have:

- $\boldsymbol{A} = \{B \subseteq X | \text{card}(B) = 3\}$, is a $C-(X, D, c)$ TOP-design of the 2nd type, but not of the 1st and 3rd types.
- $\boldsymbol{A} = \{\{0, x, 1\} | 0 < x < 1\}$, is a $C-(X, D, c)$ TOP-design of the 3rd type, but not of the 1st and 2nd types.
- $\boldsymbol{A} = \{\{0, x, 1\} | 0 < x < 1\} \cup \left\{\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}\right\}$, is a $C-(X, D, 1)$ TOP-design of the 4th type, but not of the 1st, 2nd, and 3rd types.

Therefore the diagram established in Remark 4.5 is valid and the implications are not reversible.

Note 5.3.

(1) Let X be a topological space such that for each $Y, Z \subseteq X$, there exists a homeomorphism $f : X \to X$ with $f(Y) = Z$ if and only if Y, Z are homeomorph (you may choose weaker conditions too), then all types of $C-(X, D, \lambda)$ TOP-design are the same.

(2) Considering abstract sets as discrete topological spaces, we will have usual designs as TOP-designs.

(3) Considering well-ordered sets by their order topology, we will have (well-ordered)-designs as TOP-designs.

Example 5.4. In $X = \left\{\frac{1}{n} | n \in \mathbb{N}\right\} \cup \{0\}$ with the induced topology of $\mathbb{R}$, for each nonempty subset S of X there exists a unique $T \in \{\{\frac{1}{n} | n \geq t\} \cup \{0\} | t \in \mathbb{N}\} \cup \{\{\frac{1}{n} | 1 \leq n \leq t\} \cup \{0\} | t \in \mathbb{N}\} \cup \{\{\frac{1}{n} | n \geq t\} | t \in \mathbb{N}\} \cup \{\{\frac{1}{n} | 1 \leq n \leq t\} | t \in \mathbb{N}\} \cup \{\{\frac{1}{2n} | n \in \mathbb{N}\} \cup \{0\}, \{\frac{1}{2n} | n \in \mathbb{N}\}, \{0\}\}$ such that $S \approx T$ and $X - S \approx X - T$, moreover the following table is valid (where $D_{1,t} = \{\frac{1}{n} | n \geq t\} \cup \{0\}$, $D_{2,t} = \{\frac{1}{2n} | n \in \mathbb{N}\} \cup \{0\}$, $D_{3,t} = \{\frac{1}{n} | 1 \leq n \leq t\} \cup \{0\}$, $D_{4,t} = \{0\}$, $D_{5,t} = \{\frac{1}{n} | n \geq t\}$, $D_{6,t} = \{\frac{1}{2n} | n \in \mathbb{N}\}$, and $D_{7,t} = \{\frac{1}{n} | 1 \leq n \leq t\}$):

m	n \ i	1		2		3		4	
1	1	$k>1$	---	Unique answer: $A=\{X\}$ and $\lambda=1$		$k>l$	---	Let $p\le l$, $\lambda=\binom{l-1}{l-p}$ $E=\left\{\frac{1}{n}\mid n\ge p\right\}\cup\{0\}$, and $A=\{B\subseteq X\mid B\approx E,\ X-B\approx X-E\}$	
		$k=1$	Unique answer: $A=\{X\}$ and $\lambda=1$			$k\le l$	$\lambda=\binom{l-1}{l-k}$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$		
	2	---				---			
	3	---							
	4								
	5								
	6								
	7								
2	1	$k>1$	---	Unique answer: $A=\{X\}$ and $\lambda=1$		$\lambda=\max\{(k-1)\aleph_0,1\}$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$		$\lambda=\aleph_0$, $A=\left\{X-\left\{\frac{1}{n}\right\}\mid n\in \mathrm{N}\right\}$	
		$k=1$	Unique answer: $A=\{X\}$ and $\lambda=1$						
	2	---				$\lambda=c$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$			
	3	---							
	4								
	5								
	6								
	7								
3	1	$\lambda=\max\{(k-1)\aleph_0,1\}$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$		$\lambda=\max\{(k-1)\aleph_0,1\}$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$		$\lambda=\max\{(k-1)\aleph_0,1\}$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$		$\lambda=\aleph_0$, $A=\left\{X-\left\{\frac{1}{n}\right\}\mid n\in \mathrm{N}\right\}$	
	2	$\lambda=c$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$				$\lambda=c$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$			
	3	$l\ge k-1$	---	$l>k$	---	$l>k$	---	$l>k$	---
				$l=k$	Unique answer: $\lambda=1$, $A=\{B\subseteq X\mid B\approx D\}$	$l\le k$	$\lambda=\max\{(k-l)\aleph_0,1\}$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$	$l=k$	$\lambda=1$, $A=\{B\subseteq X\mid B\approx D\}$
		$l<k-1$	$\lambda=\aleph_0$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$	$l<k$	$\lambda=\aleph_0$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$			$l<k$	$\lambda=\aleph_0$, $A=\{B\subseteq X\mid B\approx D,\ X-B\approx X-D\}$
	4	---							
	5								
	6								

Table 2

	7	---		$l > k-1$	---	---		$l > k-1$	---
				$l = k-1$	Unique answer: $\lambda = 1$, $A = \{B \subseteq X \mid B \approx D\}$			$l = k-1$	Unique answer: $\lambda = 1$, $A = \{B \subseteq X \mid B \approx D\}$
				$l < k-1$	$\lambda = \aleph_0$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$			$l < k-1$	$\lambda = \aleph_0$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$
4	1	$\lambda = \max\{(k-1)\aleph_0, 1\}$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$		$\lambda = c$, $A = \{B \subseteq X \mid B \approx D\}$		$\lambda = \max\{(k-1)\aleph_0, 1\}$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$		$\lambda = c$, $A = \{B \subseteq X \mid B \approx D\}$	
	2	$\lambda = c$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$				$\lambda = c$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$			
	3	$k > 1$	$\lambda = \aleph_0$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$	$\lambda = \aleph_0$, $A = \{B \subseteq X \mid B \approx D\}$		$\lambda = \aleph_0$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$		$\lambda = \aleph_0$, $A = \{B \subseteq X \mid B \approx D\}$	
		$k = 1$	---						
	4	---		Unique answer: $\lambda = 1$ and $A = \{\{x\} \mid x \in X\}$		Unique answer: $\lambda = 1$ and $A = \{\{0\}\}$		It is necessary and sufficient that $\{\{0\}\} \subseteq A \subseteq \{\{x\} \mid x \in X\}$ and $\lambda = 1$	
	5	---							
	6								
	7	---		$\lambda = \max\{(k-1)\aleph_0, 1\}$, $A = \{B \subseteq X \mid B \approx D\}$		---		$A = \{B \subseteq X \mid B \approx D\}$, $\lambda = \max\{(k-1)\aleph_0, 1\}$	
5	1	$k > 1$	---	Unique answer: $A = \{X\}$ and $\lambda = 1$		$k > l$	---	Let $p \le l$, $\lambda = \binom{l-1}{l-p}$ $E = \left\{\frac{1}{n} \mid n \ge p\right\} \cup \{0\}$, and $A = \{B \subseteq X \mid B \approx E,\ X - B \approx X - E\}$	
		$k = 1$	Unique answer: $A = \{X\}$ and $\lambda = 1$			$k \le l$	$\lambda = \binom{l-1}{l-k}$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$		
	2	---				---			
	3	---							
	4								
	5	$k > 1$	---	Unique answer: $A = \{X - \{0\}\}$ and $\lambda = 1$		$k > l$	---	Let $p \le l$, $\lambda = \binom{l-1}{l-p}$ $E = \left\{\frac{1}{n} \mid n \ge p\right\} \cup \{0\}$, and $A = \{B \subseteq X \mid B \approx E,\ X - B \approx X - E\}$	
		$k = 1$	Unique answer: $A = \{X\}$ and $\lambda = 1$			$k \le l$	$\lambda = \binom{l-1}{l-k}$, $A = \{B \subseteq X \mid B \approx D,\ X - B \approx X - D\}$		
	6	---				---			
	7	---							

Table 2 (cont.)

6	1	$k>1$	---	Unique answer: $A=\{X\}$ and $\lambda=1$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D\}$	
		$k=1$	Unique answer: $A=\{X\}$ and $\lambda=1$						
	2	---				$\lambda=c$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$			
	3	---							
	4								
	5	$k>1$	---	Unique answer: $A=\{X-\{0\}\}$ and $\lambda=1$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D\}$	
		$k=1$	Unique answer: $A=\{X-\{0\}\}$, $\lambda=1$						
	6	---							
	7	---							
7	1	$\lambda=\max\{(k-1)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D\}$		$\lambda=\max\{(k-1)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D\}$	
	2	$\lambda=c$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$				$\lambda=c$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$			
	3	$l\geq k$	---	$l>k+1$	---	$l>k$	---	$l>k+1$	---
		$l<k$	$\lambda=\aleph_0$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$	$l\leq k+1$	$\lambda=\max\{(k+1-l)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D\}$	$l\leq k$	$\lambda=\max\{(k+1-l)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$	$l\leq k+1$	$\lambda=\max\{(k+1-l)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D\}$
	4	---		$l>1$	---	---		$l>1$	---
				$l=1$	Unique answer: $\lambda=1$, $A=\{\{x\} \mid x\in X\}$			$l=1$	It is necessary and sufficient that $\{\{0\}\}\subseteq A\subseteq\{\{x\} \mid x\in X\}$ and $\lambda=1$
	5	---				$\lambda=\aleph_0$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$		$\lambda=c$, $A=\{B\subseteq X \mid B\approx D\}$	
	6	---				$\lambda=c$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$			
	7	---		$l>k$	---	$l>k$	---	$l>k$	---
				$l=k$	Unique answer: $\lambda=1$, $A=\{B\subseteq X \mid B\approx D\}$	$l\leq k$	$\lambda=\max\{(k-l)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$	$l\leq k$	$\lambda=\max\{(k-l)\aleph_0,1\}$, $A=\{B\subseteq X \mid B\approx D,\ X-B\approx X-D\}$
				$l<k$	$\lambda=\aleph_0$, $A=\{B\subseteq X \mid B\approx D\}$				

Table 2 (cont.)

In the above table for $X = \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$, $C = D_{m,l}$, and $D = D_{n,k}$. A gray box indicates that there is no $C - (X, D, \lambda)$ TOP-design of type i, and the statement written in a no-color box presents one of the possible $C - (X, D, \lambda)$ TOP-design of type i (or sometimes the unique possibility) named $\boldsymbol{A}$.

Example 5.5. In a metric space, we will use the concept of isometric subspaces (although a metric space is a topological space too and we are able to use the concept of homeomorphism, but this latter case is left for TOP-design, and we will search for metric-design).

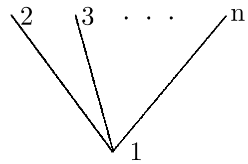
- In finite metric space (X, d), where:

$$d(x,y) = \begin{cases} 1 & x \neq y, \\ 0 & x = y, \end{cases}$$

  for each $E, F \subseteq X$, E and F are isometric if and only if $\text{card}(E) = \text{card}(F)$. For $C, D \subseteq X$, $\boldsymbol{A}$ is a $C - (X, D, \lambda)$ metric-design of any type if and only if $\boldsymbol{A}$ is a $\text{card}(C) - (X, \text{card}(D), \lambda)$ design.
- In finite metric space (X, d), with $X = \{1, , n\}$ and the induced metric from $\mathbb{R}$. We have:
  - for each $C, D \subseteq X$, $\boldsymbol{A}$ is a $C - (X, D, \lambda)$ metric-design of the 1st type if and only if $\{1, n\} \subseteq C \subseteq D$, $\lambda = 1$, and $\boldsymbol{A} = \{D\}$;
  - for each $C, D \subseteq X$, $\boldsymbol{A}$ is a $C - (X, D, \lambda)$ metric-design of the 3rd type if and only if $C \subseteq D$, $\lambda = 1$, and $\boldsymbol{A} = \{D\}$;
  - for each $C, D \subseteq X$, $\boldsymbol{A}$ is a $C - (X, D, \lambda)$ metric-design of the 4th type if and only if $\boldsymbol{A} \subseteq \{K \subseteq X | K \text{ and } D \text{ are isometric}\}$, $\lambda = 1$, and there exists $E \in \boldsymbol{A}$ such that $C \subseteq E$.
- In the two examples above we have the same TOP-designs but different metric-designs for two reference sets.

Example 5.6. In a graph $G = (V, E)$, for $A \subseteq V$ by $G(A)$ we mean induced subgraph from G on A and $G - G(A)$ means $G(V - A)$. In this example by A (when A is a subset of V), we mean $G(A)$. Now consider the following examples:

- If G is a finite complete graph on n vertices and $C, D \subseteq V$, then there exists a $C - (V, D, \lambda)$ graph-design (indeed (induced-graph)-design) of any type if and only if there exists a $\text{card}(C) - (\text{card}(V), \text{card}(D), \lambda)$ design.
- If G is the following complete bipartite graph, for $n \geq 3$:



Let (for each $B \subseteq V$, there exists a unique $k \in \{1, ..., 2n-1\}$ such that $B \approx A_k$ and $V - B \approx V - A_k$ (i.e., induced subgraphs by B and A_k are isomorph, and induced subgraphs by $V - B$ and $V - A_k$ are isomorph too)):

$$A_k = \begin{cases} \{1, ..., k\} & 1 \leq k \leq n, \\ \{2, ..., k+1-n\} & n+1 \leq k \leq 2n-1. \end{cases}$$

We have the following table.

i	j \ h	1	2	3	4
1	1		Unique answer: $A=\{\{1\},\dots,\{n\}\},\lambda=1$	Unique answer: $A=\{\{1\}\},\lambda=1$	2^{n-1} answer: $A\in\{\{\{1\}\}\cup D: D\subseteq\{\{2\},\dots,\{n\}\}\},\ \lambda=1$
	$2,\dots,n-1$			$\binom{n-1}{j-1}$ answer: A is a nonempty subset of $\{\{1,p_1,\dots,p_{j-1}\}:1<p_1<\dots<p_{j-1}\le n\}\},\ \lambda=\text{card}(A)$	
	n	Unique answer: $A=\{\{1,\dots,n\}\},\lambda=1$			
	$n+1$		Unique answer: $A=\{\{1\},\dots,\{n\}\},\lambda=1$		2^{n-1} answer: $A\in\{\{\{1\}\}\cup D: D\subseteq\{\{2\},\dots,\{n\}\}\},\ \lambda=1$
	$n+2,\dots,2n-1$				
$2,\dots,n$	$1,\dots,j-1$				
	$i,\dots,n$	A is an $A_i-(V,A_j,\lambda)$ graph-design iff A is a nonempty subset of $\{\{1,p_1,\dots,p_{j-1}\}:1<p_1<\dots<p_{j-1}\le n\}\}$ and $\{\{p_1,\dots,p_{j-1}\}:\{1,p_1,\dots,p_{j-1}\}\in A\}$ is an $(i-1)-(n-1,j-1,\lambda)$ design			
	$n+2,\dots,2n-1$				
$n+1$	1		Unique answer: $A=\{\{1\},\dots,\{n\}\},\lambda=1$		Just two answers: $A=\{\{1\},\dots,\{n\}\},\{\{2\},\dots,\{n\}\},\lambda=1$
	$2,\dots,n-1$			A is an $A_i-(V,A_j,\lambda)$ graph-design iff A is a nonempty subset of $\{\{1,p_1,\dots,p_{j-1}\}:1<p_1<\dots<p_{j-1}\le n\}\}$ and $\{\{p_1,\dots,p_{j-1}\}:\{1,p_1,\dots,p_{j-1}\}\in A\}$ is a $1-(n-1,j-1,\lambda)$ design	
	n	Unique answer: $A=\{\{1,\dots,n\}\},\lambda=1$			
	$n+1$		Unique answer: $A=\{\{1\},\dots,\{n\}\},\lambda=1$	Unique answer: $A=\{\{2\},\dots,\{n\}\},\lambda=1$	Just two answers: $A=\{\{1\},\dots,\{n\}\},\{\{2\},\dots,\{n\}\},\lambda=1$
	$n+2,\dots,2n-1$			A is an $A_i-(V,A_j,\lambda)$ graph-design iff A is a nonempty subset of $\{\{p_1,\dots,p_{j-n}\}:1<p_1<\dots<p_{j-n}\le n\}\}$ and $\{\{p_1,\dots,p_{j-n}\}:\{p_1,\dots,p_{j-n}\}\in A\}$ is a $1-(n-1,j-n,\lambda)$ design	
$n+2,\dots,2n-1$	$1,\dots,i-n$				
	$i-n+1,\dots,n$	A is an $A_i-(V,A_j,\lambda)$ graph-design iff A is a nonempty subset of $\{\{1,p_1,\dots,p_{j-1}\}:1<p_1<\dots<p_{j-1}\le n\}\}$ and $\{\{p_1,\dots,p_{j-1}\}:\{1,p_1,\dots,p_{j-1}\}\in A\}$ is an $(i-n)-(n-1,j-1,\lambda)$ design			
	$n+1,\dots,j-1$				
	$i,\dots,2n-1$	A is an $A_i-(V,A_j,\lambda)$ graph-design iff A is a nonempty subset of $\{\{p_1,\dots,p_{j-n}\}:1<p_1<\dots<p_{j-n}\le n\}\}$ and $\{\{p_1,\dots,p_{j-n}\}:\{p_1,\dots,p_{j-n}\}\in A\}$ is an $(i-n)-(n-1,j-n,\lambda)$ design			

Table 3

In the above table about the existence of an $A_i-(V,A_j,\lambda)$ graph-design (indeed (induced-subgraph)-design) named A of the hth type we have the corresponding statement, moreover a (gray box) indicates that there isn't any $A_i-(V,A_j,\lambda)$ graph-design in the corresponding case.

Example 5.7. In a measure space (X, Σ, μ), a function $f : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is called measurable if for each $M \in \Sigma$ we have $f^{-1}(M) \in \Sigma$. A measurable function $f : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is called measure preserving if $\mu(f^{-1}(M)) = \mu(M)$ for each $M \in \Sigma$. Bijection $f : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is called invertible measure preserving if both $f, f^{-1} : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ are measure preserving [9]. In a measure space (X, Σ, μ), a $C - (X, D, \lambda)$ measure-design is considered just for $C, D \in \Sigma$. Consider the following examples.

- If X is finite, Σ is the collection of all subsets of X, and μ is counting measure on X, then for each nonempty subsets C, D of X, $\boldsymbol{A}$ is a $C - (X, D, \lambda)$ measure-design (of any type) if and only if $\boldsymbol{A}$ is a $\mathrm{card}(C) - (X, \mathrm{card}(D), \lambda)$ design.
- If μ is dirac measure on X for $a \in X$ and Σ is the collection of all subsets of X, thus:

$$\mu(E) = \begin{cases} 1 & a \in E, \\ 0 & a \notin E, \end{cases}$$

  and for each $E, F \subseteq X$ there exists an invertible measure preserving function from E to F if and only if $E \cap \{a\} = F \cap \{a\}$ and $\mathrm{card}(E) = \mathrm{card}(F)$. Therefore for each nonempty subsets C, D of X, $\boldsymbol{A}$ is a $C - (X, D, \lambda)$ measure-design of the ith type if and only if one of the following conditions holds:
  - for each $E \in \boldsymbol{A}$, $E \cap \{a\} = D \cap \{a\} = \{a\}$, $C, D \neq \{a\}$ and $\{E - \{a\} : E \in \boldsymbol{A}$ is a $(C - \{a\}) - (X - \{a\}, D - \{a\}, \lambda)$ set design of type i,
  - for each $E \in \boldsymbol{A}$, $E \cap \{a\} = D \cap \{a\} = C \cap \{a\}$ is the empty set and $\boldsymbol{A}$ is a $C - (X - \{a\}, D, \lambda)$ set design of type i,
  - for each $E \in \boldsymbol{A}$, $E \cap \{a\} = D \cap \{a\} = C = \{a\}$, and $\lambda = \mathrm{card}(\boldsymbol{A})$.

Note 5.8. One may consider more than one specified mathematical structure like being a measure space and a poset, and consequently considered the related combined topic for a generalized-design.

Theorem 5.9. In a totally-ordered set $(W, \leq)$, if $\boldsymbol{A}_1$ is a $C_1 - (S, D_1, \lambda_1)$ set-design of the 2nd type and $\boldsymbol{A}_2$ is a $C_2 - (W, D_2, \lambda_2)$ (totally-ordered)-design of the 2nd type, and $S \times W$ is considered under partial order relation $\preceq := \{((u, v), (z, w)) : u = z, v \leq w\}$, then $\boldsymbol{A} := \{B_1 \times B_2 | B_1 \in \boldsymbol{A}_1, B_2 \in \boldsymbol{A}_2\}$ is a $C_1 \times C_2 - (S \times W, D_1 \times D_2, \lambda_1\lambda_2)$ poset-design of the 2nd type.

Proof. If $K \subseteq S \times W$ and $\varphi : C_1 \times C_2 \to K$ and π_i is the projection map on the ith coordinate, choose $c_0 \in C_1$ and $d_0 \in C_2$. We have the following steps:
STEP 1. $\varphi_1 : C_1 \to \pi_1(K)$ $(\varphi_1(c) = \pi_1\varphi(c, d_0), c \in C_1)$ is bijective. If

$c, e \in C_1$ and $\varphi_1(c) = \varphi_1(e)$, since W is totally-ordered, thus $\pi_2\varphi(c, d_0) \leq \pi_2\varphi(e, d_0)$ or $\pi_2\varphi(e, d_0) \leq \pi_2\varphi(c, d_0)$, suppose $\pi_2\varphi(c, d_0) \leq \pi_2\varphi(e, d_0)$; therefore:

$$\varphi(c, d_0) = (\pi_1\varphi(c, d_0), \pi_2\varphi(c, d_0)) \preceq (\pi_1\varphi(e, d_0), \pi_2\varphi(e, d_0)) = \varphi(e, d_0)$$

and $(c, d_0) \preceq (e, d_0)$ which leads to $c = e$. For $u \in \pi_1(K)$, there exists $(c, d) \in K$ such that $u = \pi_1\varphi(c, d)$, since W is totally-ordered, thus $d \leq d_0$ or $d_0 \leq d$, suppose $d \leq d_0$, thus $(c, d) \preceq (c, d_0)$ and $\varphi(c, d) \preceq \varphi(c, d_0)$ which leads to $u = \pi_1\varphi(c, d) = \pi_1\varphi(c, d_0) = \varphi_1(c)$.
STEP 2. $\varphi_2 : C_2 \to \pi_2(K)$ $(\varphi_2(d) = \pi_2\varphi(c_0, d), d \in C_2)$ is an order isomorphism, moreover for $B_1 \in \boldsymbol{A}_1$, $B_2 \in \boldsymbol{A}_2$, $K \subseteq B_1 \times B_2$ if and only if $\pi_1(K) \times \pi_2(K) \subseteq B_1 \times B_2$. It is evident that

$$\begin{aligned} &\text{card}(\{B_1 \times B_2 \in \boldsymbol{A} | K \subseteq B_1 \times B_2\}) \\ = \quad &\text{card}(\{B_1 \times B_2 \in \boldsymbol{A} | \pi_1(K) \times \pi_2(K) \subseteq B_1 \times B_2\}) \\ = \quad &\text{card}(\{B_1 \times B_2 | B_1 \in \boldsymbol{A}_1, B_2 \in \boldsymbol{A}_2, \pi_1(K) \subseteq B_1, \pi_2(K) \subseteq B_2\}) \\ = \quad &\text{card}(\{B_1 \in \boldsymbol{A}_1 | \pi_1(K) \subseteq B_1\})\text{card}(\{B_2 \in \boldsymbol{A}_2 | \pi_2(K) \subseteq B_2\}) = \lambda_1\lambda_2, \end{aligned}$$

which completes the proof. □

Theorem 5.10. In partial ordered sets $(P_1, \leq_1)$ and $(P_2, \leq_2)$, if $\boldsymbol{A}_1$ is a $C_1 - (P_1, D_1, \lambda_1)$ poset-design of the 2nd type and $\boldsymbol{A}_2$ is a $C_2 - (P_2, D_2, \lambda_2)$ poset-design of the 2nd type, $P_1 \cap P_2 = \emptyset$, and $P_1 \cup P_2$ is considered under partial order relation $\leq_1 \cup \leq_2$ such that for each $K \subseteq P_1 \cup P_2$, K is isomorphic to $C_1 \cup C_2$ if and only if $K \cap P_i$ is isomorphic to C_i for $i = 1, 2$, then $\boldsymbol{A} := \{B_1 \cup B_2 : B_1 \in \boldsymbol{A}_1, B_2 \in \boldsymbol{A}_2\}$ is a $C_1 \cup C_2 - (P_1 \cup P_2, D_1 \cup D_2, \lambda_1\lambda_2)$ poset-design 2nd type.

Proof. It is clear. □

Note. If $\leq_1 = \{(x, x) : x \in P_1\}$, $(P_2, \leq_2)$ is a totally ordered set, $P_1 \cap P_2 = \emptyset$, and C_2 has more than one element, then the assumptions of this theorem are satisfied.

Theorem 5.11. For $i = 1, ..., n$, let $\boldsymbol{A}_i$ be a $C_i - (S_i, D_i, \lambda_i)$ set-design of the 2nd type, and S_i's be disjoint, then $\boldsymbol{A} := \left\{\bigcup_{i=1}^{n} B_i | B_i \in \boldsymbol{A}_i\right\}$ is a $\bigcup_{i=1}^{n} C_i - \left(\bigcup_{i=1}^{n} S_i, \bigcup_{i=1}^{n} D_i, \lambda_1 \cdots \lambda_n\right)$ poset-design of the 2nd type, where $S := \bigcup_{i=1}^{n} S_i$ is considered under relation $\leq := \{(a, b) \in S \times S | a = b \vee \exists i < j(a \in S_i \wedge b \in S_j)\}$ $(n \in \mathbb{N})$.

Proof. Let for each $i \in \{1,...,n\}$, $B_i \in \boldsymbol{A}_i$ and $\varphi_i : B_i \to D_i$ be a bijection, then $\bigcup_{i=1}^{n} \varphi_i : \bigcup_{i=1}^{n} B_i \to \bigcup_{i=1}^{n} D_i$ is an order isomorphism (with induced relation of), moreover if each $i \in \{1,...,n\}$, K_i and C_i is equipotent, then $\bigcup_{i=1}^{n} K_i$ and $\bigcup_{i=1}^{n} C_i$ are order isomorphic. Now suppose $K \subseteq S$ and $\varphi : \bigcup_{i=1}^{n} C_i \to K$ be an order isomorphism, for $i \in \{1,...,n\}$ choose $c_i \in C_i$, since $\varphi(c_1) < \cdots < \varphi(c_n)$ thus $\varphi(c_i) \in S_i$ (for each chain in S of length n, like L, $L \cap S_i$ has exactly one element), therefore $\varphi|_{C_i} : C_i \to K \cap S_i$ is a bijection. Therefore for each $K \subseteq S$, K and $\bigcup_{i=1}^{n} C_i$ are order isomorphic if and only if for each $i \in \{1,...,n\}$, $K \cap S_i$ and C_i are equipotent. Thus for $K \subseteq S$ such that K and $\bigcup_{i=1}^{n} C_i$ are order isomorphic, we have:

$$\begin{aligned}
&\operatorname{card}(\{B \in \boldsymbol{A} | K \subseteq B\}) \\
= \;& \operatorname{card}(\{B \in \boldsymbol{A} | \forall i \in \{1,...,n\} \quad K \cap S_i \subseteq B \cap S_i\}) \\
= \;& \operatorname{card}(\{(B_1,...,B_n) \in \boldsymbol{A}_1 \times \cdots \boldsymbol{A}_n | \forall i \in \{1,...,n\} \quad K \cap S_i \subseteq B_i\}) \\
= \;& \operatorname{card}(\{B_1 \in \boldsymbol{A}_1 | K \cap S_1 \subseteq B_1\} \times \cdots \times \{B_n \in \boldsymbol{A}_n | K \cap S_n \subseteq B_n\}) \\
= \;& \lambda_1 \cdots \lambda_n
\end{aligned}$$

□

Theorem 5.12. If $(\Gamma, \preceq)$ is a nonempty totally-ordered set such that for each $\alpha \in \Gamma$, $\boldsymbol{A}_\alpha$ is a $C_\alpha - (S_\alpha, D_\alpha, \lambda)$ set-design of the 2nd type, C_α's are equipotent, D_α's are equipotent too, and S_α's are disjoint, then $\boldsymbol{A} := \bigcup_{\alpha \in \Gamma} \boldsymbol{A}_\alpha$ is a $C_{\alpha_0} - \left(\bigcup_{\alpha \in \Gamma} S_\alpha, D_{\alpha_0}, \lambda \right)$ poset-design of the 2nd type, where $\alpha_0 \in \Gamma$ is arbitrary and $S := \bigcup_{\alpha \in \Gamma} S_\alpha$ is considered under partial order relation $\leq := \{(a,b) \in S \times S | a = b \vee \exists \alpha \prec \beta (a \in S_\alpha \wedge b \in S_\beta)\}$.

Proof. Use the fact that $K(\subseteq S)$ is order isomorphic with C_{α_0} if and only if there exists $\alpha \in \Gamma$ such that $K \subseteq S_\alpha$ and K is equipotent with C_α. □

Theorem 5.13. If $(\Gamma, \preceq)$ is a nonempty totally-ordered set such that for each $\alpha \in \Gamma$, $\boldsymbol{A}_\alpha$ is a $C_\alpha - (S_\alpha, D_\alpha, \lambda)$ set-design of the 2nd type, C_α's are equipotent, $\boldsymbol{A}_\alpha$'s are equipotent, and S_α's are disjoint, then $\boldsymbol{A} :=$

$\left\{\bigcup_{\alpha\in\Gamma} B_\alpha | \forall\alpha\in\Gamma \;\; B_\alpha\in \boldsymbol{A}_\alpha\right\}$ is a $C_{\alpha_0} - \left(\bigcup_{\alpha\in\Gamma} S_\alpha, \bigcup_{\alpha\in\Gamma} D_\alpha, \lambda\mu^{\gamma-1}\right)$ poset-design of the 2nd type, where $\alpha_0 \in \Gamma$ is arbitrary, $\gamma = \text{card}(\Gamma)$, $\mu = \text{card}(\boldsymbol{A}_\alpha)$ and $S := \bigcup_{\alpha\in\Gamma} S_\alpha$ is considered under $\leq := \{(a,b) \in S \times S | a = b \vee \exists\alpha \prec \beta(a \in S_\alpha \wedge b \in S_\beta)\}$.

Proof. Use a similar method described in Theorem 5.12. □

Theorem 5.14. If $(\Gamma, \preceq)$ is an infinite totally-ordered set such that for each $\alpha \in \Gamma$, $\boldsymbol{A}_\alpha$ is a $C_\alpha - (S_\alpha, D_\alpha, \lambda_\alpha)$ set-design of the 2nd type, there exists $\beta \in \Gamma - \{\alpha\}$ with $\text{card}(\boldsymbol{A}_\alpha) \leq \text{card}(\boldsymbol{A}_\beta)$, and S_α's are disjoint, then $\boldsymbol{A} := \{\bigcup_{\alpha\in\Gamma} B_\alpha | \forall\alpha\in\Gamma \quad B_\alpha \in \boldsymbol{A}_\alpha\}$ is a $C_{\alpha_0} - \left(\bigcup_{\alpha\in\Gamma} S_\alpha, \bigcup_{\alpha\in\Gamma} D_\alpha, \mu\right)$ poset-design of the 2nd type, where $\alpha_0 \in \Gamma$ is arbitrary, $\gamma = \prod_{\alpha\in\Gamma} \text{card}(\boldsymbol{A}_\alpha)$ $\left(= \text{card}\left(\prod_{\alpha\in\Gamma} \boldsymbol{A}_\alpha\right)\right)$, and $S := \bigcup_{\alpha\in\Gamma} S_\alpha$ is considered under partial order relation $\leq := \{(a,b) \in S \times S | a = b \vee \exists\alpha \prec \beta(a \in S_\alpha \wedge b \in S_\beta)\}$.

Proof. Use a similar method described in Theorem 5.12. Moreover note the fact that since Γ is infinite and for each $\alpha \in \Gamma$ there exists $\beta \in \Gamma - \{\alpha\}$ such that $\text{card}(\boldsymbol{A}_\alpha) \leq \text{card}(\boldsymbol{A}_\beta)$, and $\lambda_\alpha > 0$, thus for each $\theta \in \Gamma$ we have $\lambda_\theta \prod_{\alpha\in\Gamma-\{\theta\}} \text{card}(\boldsymbol{A}_\alpha) = \prod_{\alpha\in\Gamma} \text{card}(\boldsymbol{A}_\alpha)$. (For more details on infinite products of cardinals we refer the interested reader to [7, Section 1.6]). □

6. Conclusion

In this text we defined four generalizations of designs for some mathematical structures $\mathcal{M}$ like TOP-designs, (well-ordered)-designs, etc. in which if X has structure $\mathcal{M}$, then for each $A \subseteq X$, A has structure $\mathcal{M}$ as well. The reader may find it interesting to consider theses generalizations for other structures like semigroup-designs, group-designs, or ring-designs, but some problems may arise. For example, all of the subsets of a group do not have group structure, so we suggest using subgroups generated (see Examples 5.6 and 5.7). In this way we will have a new scope. Sometimes the notion of generalized design refers to H-design H-(v, g, κ, t) (a generalization of Steiner systems). H-design is a triple (X, G, B), where X is a set of points whose cardinality is vg, and $G = \{G_1, \ldots, G_v\}$ is a partition of X into v sets of cardinality g; the members of G are called groups, a transverse of G is a subset of X that meets each group in at most one point, the set

B contains $\kappa-$element transverse of G, called blocks, with the property that each $t-$element transverse of G is contained in precisely one block (first introduced in [6] and noted in [8]); but we have a different sense of generalization.

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