BLIND MEN AND HYPERCUBES

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Abstract. In this paper an elementary probability question is solved and the procedure used is generalized to higher dimensions.

1. Introduction

In [1] Díaz-Barrero posed the following question: Suppose we have a cube with edge equal to n, built of n^3 white cubes of edge one. The surface of the cube is painted black. A blind man splits the cube. What is the probability that he will be able to assemble a cube that looks like the original cube, in the sense that the all the black faces are outside? The problem of assembling the cube is in two parts: (1) for each position, pick a small hypercube with the correct coloring; (2) having picked it, orient it correctly so that the black faces are outside. The answer to the preceding question is that the probability asked is very small. In this paper this probability is computed and we generalize the procedure to general structures in higher dimensions.

2. Main Results

In what follows we compute explicitly the probability asked in [1] and some generalizations of this problem are also given. We begin with the following theorem.

Theorem 1. *The surface of a cube with edge equal to n, built of n^3 white cubes of edge one is painted black. If a blind man splits the cube, then the probability that he assembles a cube that looks like the original one is*

$$P(3;n) = \frac{8!}{8^8} \cdot \frac{(12\,(n-2))!}{12^{12(n-2)}} \cdot \frac{\big(6(n-2)^2\big)!}{6^{6(n-2)^2}} \cdot \frac{(n-2)^3!}{n^3!}.$$

Proof. We observe that the set of small cubes can be partitioned into some subsets where the cubes are grouped according to the number of black faces. For $0 \le i \le 3$, let A_i be the set of small cubes having i painted faces. It is clear that the A_i's are nonempty sets pairwise disjoint and they form a partition of the set of all small cubes with cardinal $|A_3| = 8$, $|A_2| = 12(n-2)$, $|A_1| = 6(n-2)^2$, and $|A_0| = (n-2)^3$, respectively.

Let us denote by $P(3;n)$ the probability of the event that all the small cubes be correctly oriented and let $P(A_i)$ be the probability that A_i fits correctly in the original position ($0 \le i \le 3$). Then, applying the multiplicative probability rule, we have

$$P(3;n) = P(A_3) \cdot P(A_2|A_3) \cdot P(A_1|A_3 \cap A_2) \cdot P(A_0|A_3 \cap A_2 \cap A_1).$$

Now we will compute the probabilities. We compute first $P(A_3)$. The probability to take the first element from A_3 is $\frac{8}{n^3}$. We also have to count the probability that this element of A_3 fits into its original place. This small cube has to have its three black colored faces placed in the corner of the cube. So this probability is $\frac{1}{8}$ because the element has eight corners, each equally likely to be placed in the position where the black corner should go; and the probability that the first element fits correctly is $\frac{1}{8} \cdot \frac{8}{n^3}$. For the second one we have $\frac{1}{8} \cdot \frac{7}{n^3-1}$, and so on until the probability that the last element of A_3 to be taken is $\frac{1}{8} \cdot \frac{1}{n^3-7}$. Therefore,

$$P(A_3) = \frac{8!}{n^3(n^3-1)\cdots(n^3-7)} \left(\frac{1}{8}\right)^8.$$

To compute $P(A_2|A_3)$, we have to calculate the probability that all elements of A_2 be correctly placed and that each element be correctly oriented. Taking into account that a cube has 12 edges, the probability that a certain element be correctly placed and correctly oriented is now $\frac{1}{12}$, because it has to fit with the colored edge. Thus,

$$P(A_2|A_3) = \frac{[12(n-2)]!}{(n^3-8)\cdots[n^3-8-12(n-2)+1]} \left(\frac{1}{12}\right)^{12(n-2)}.$$

Likewise for $P(A_1|A_3 \cap A_2)$, taking into account that the cube has 6 faces, we have

$$P(A_1|A_3 \cap A_2) =$$

$$\frac{[6(n-2)^2]!}{[n^3-8-12(n-2)]\cdots[n^3-6(n-2)^2-12(n-2)-8+1]} \left(\frac{1}{6}\right)^{6(n-2)^2}.$$

Finally,

$$P(A_0|A_3 \cap A_2 \cap A_1) = \frac{(n-2)^3!}{(n-2)^3!} = 1$$

and

$$P(3;n) =$$

$$\frac{8!\,[12(n-2)]! \cdot [6(n-2)^2]!\,(n-2)^3!}{n^3!} \left(\frac{1}{8}\right)^8 \left(\frac{1}{12}\right)^{12(n-2)} \left(\frac{1}{6}\right)^{6(n-2)^2},$$

from which the statement immediately follows and the proof is complete. ∎

Next, we state and prove a result that we will use later.

Theorem 2. *Let $f_{d,i}$ be the number of faces with dimension i in a hypercube in d-dimensions. Then $f_{d,i} = \binom{d}{i} 2^{d-i}$.*

Proof. Since the d–hypercube is obtained from the $(d-1)$–hypercube, adding the translation over e_d, the vector of the canonical base, then we have the relation

$$f_{d,i} = 2f_{d-1,i} + f_{d-1,i-1}.$$

This is because in the d–hypercube we can obtain faces of dimension i in two ways: (i) each face of dimension i from the $(d-1)$–hypercube gets a pair (the original face and its translate); (ii) each face of dimension $i-1$ in the $(d-1)$–hypercube, adding the e_d direction, increases its dimension by one so it generates a face of dimension i in the d–hypercube.

Now we prove that the above recurrence leads us to the mentioned relation. We proceed by mathematical induction with respect to d. Clearly for $d = 1$ (the line segment), $f_{1,1} = 1$ and $f_{1,0} = 2$, which correspond to our formula. Clearly,

$$2f_{d-1,i} + f_{d-1,i-1} = 2\binom{d-1}{i} 2^{d-1-i} + \binom{d-1}{i-1} 2^{d-i} = \binom{d}{i} 2^{d-i} = f_{d,i}.$$

In the preceding, we have used that $\binom{d}{i} = \binom{d-1}{i} + \binom{d-1}{i-1}$.

Notice that Euler's identity for convex polytopes clearly holds for hypercubes [2, 3]. Namely, if P is a nonempty polytope of dimension d having f_0 vertices, f_1 edges, $\ldots$, and f_{d-1} faces, then

$$f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d. \tag{2.1}$$

In our case $f_i = \binom{d}{i} 2^{d-i}$ and we have

$$\sum_{i=0}^{d-1} (-1)^i \binom{d}{i} 2^{d-i} = (2-1)^d - (-1)^d,$$

so (2.1) holds. ∎

Later on in this paper, when the dimension d is fixed, we will use the notation f_i instead of $f_{d,i}$. Note that, after Theorem 2, $P(3;n)$ can also be written in the most convenient form

$$P(3;n) = \frac{|A_3|!\,|A_2|!\,|A_1|!\,|A_0|!}{n^3!} \left(\frac{1}{f_0}\right)^{|A_3|} \left(\frac{1}{f_1}\right)^{|A_2|} \left(\frac{1}{f_2}\right)^{|A_1|} \left(\frac{1}{f_3}\right)^{|A_0|},$$

where f_i denote the number of elements of dimension i, $(0 \leq i \leq 3)$. For the cube, we have $f_0 = 8$, $f_1 = 12$, $f_2 = 6$, and $f_3 = 1$.

We close this section generalizing the problem of the cube to hypercubes.

Theorem 3. *Let d be a positive integer. Consider a hypercube of dimension d, with the edge equal to n, made up by n^d white unit hypercubes of dimension d. If we apply to it the same procedure as to the cube in Theorem 1, then $|A_i| = f_{d-i}\,(n-2)^{d-i}$ and*

$$P(d;n) = \frac{1}{n^d!}\prod_{0\le i\le d}|A_i|!\prod_{0\le i\le d}\left(\frac{1}{f_{d-i}}\right)^{|A_i|},$$

where $f_i = \binom{d}{i}2^{d-i}$ and $|A_i|$ is the size of the set A_i.

Proof. We observe that the set of small hypercubes can be partitioned into some subsets where the cubes are grouped according to the number of colored faces. For $0 \le i \le d$, let A_i be the set of small cubes having i painted faces. It is clear that the A_i's are nonempty sets pairwise disjoint and they form a partition of the set of all small cubes. It is easy to observe that a cube from A_i, is contained in a face of dimension $(d-i)$ (containing the intersection of i $(d-1)$–dimensional surfaces). This shows that $|A_i| = f_{d-i}\,(n-2)^{d-i}$, where as we have seen in the previous theorem, $f_i = \binom{d}{i}2^{d-i}$.

Let us denote by $P(d;n)$ the probability of the event that all the small cubes fit correctly into their original place and let $P(A_i)$ be the probability that A_i be correctly oriented $(0 \le i \le d)$. Then, applying the multiplicative probability rule, we have

$$P(d;n) =$$
$$P(A_d)\cdot P(A_{d-1}|A_d)\cdot P(A_{d-2}|A_d\cap A_{d-1})\cdots P(A_0|A_d\cap A_{d-1}\ldots\cap A_1).$$

Now we will compute the probabilities. We compute first $P(A_d)$. The probability to take the first element from A_d is $\frac{2^d}{n^d}$. We also have to count the probability that this element of A_d is correctly oriented. This small cube has to have its d black colored $(d-1)$–faces (hyper-planes) placed in the corner of the d–hypercube. So this probability is $\frac{1}{2^d}$ because the element has 2^d corners, each equally likely to be placed in the position where the black corner should go; and the probability that the first element fits correctly is $\frac{1}{2^d}\cdot\frac{2^d}{n^d}$. For the second one we have $\frac{1}{2^d}\cdot\frac{2^d-1}{n^d-1}$, and so on until the probability that the last element of A_d to be taken is $\frac{1}{2^d}\,\frac{1}{n^d-2^d+1}$. Therefore,

$$P(A_d) = \frac{2^d!}{n^d(n^d-1)\cdots(n^d-2^d+1)}\left(\frac{1}{2^d}\right)^{2^d}.$$

Observing that in this case we have $|A_d| = f_0 = 2^d$, we may write

$$P(A_d) = \frac{A_d!}{n^d(n^d-1)\cdots(n^d-|A_d|+1)}\left(\frac{1}{f_0}\right)^{|A_d|}.$$

Now we compute $P(A_{d-1}|A_d)$. We fit the elements of A_{d-1} into their previous positions, and taking into account that a cube has f_1 1-dimensional faces, the probability that a certain element of A_{d-1} fits into its position is now $\frac{1}{f_1}$, because it has to fit with the colored $(d-1)$–faces.

Thus,

$$P(A_{d-1}|A_d) = \frac{|A_{d-1}|!}{(n^d-2^d)\cdots[n^d-|A_d|-|A_{d-1}|+1]}\left(\frac{1}{f_1}\right)^{A_{d-1}}.$$

Finally, we obtain the formula

$$P(d;n) = \frac{1}{n^d!}\prod_{0\le i\le d}|A_i|!\prod_{0\le i\le d}\left(\frac{1}{f_{d-i}}\right)^{|A_i|}.$$

We just have to compute f_i and $|A_i|$, $0 \le i \le d$, for a hypercube of dimension d. □

Remark 1. *One can easily check that*

$$\sum_{i=0}^{d}|A_i| = \sum_{i=0}^{d}\binom{d}{i}2^i(n-2)^{d-i} = n^d.$$

Remark 2. *It is clear that in Theorem* 1 *and in Theorem* 3 *there is no problem if* $n = 2$*, because then* $(n-2)! = 1$*, and the results still hold in the same form. Clearly, if* $n = 1$ *the probability is* 1*.*

3. Related Results

A more general problem is to consider from the beginning a structure in two dimensions having the sides built of m, n squares and in three dimensions having the sides made of m, n, p cubes or, in general, in higher dimensions. We begin with the following theorem.

Theorem 4. *Suppose we have a box built with cubes of face* 1*, having the sides of length* m*,* n*, and* p*, respectively. If after painting the faces someone splits the box, then the probability that a blind person assembles a structure that looks like the original one is*

$$P(3;m,n,p) = \frac{|A_3|!\cdot|A_2|!\cdot|A_1|!\cdot|A_0|!}{(mnp)!}\left(\frac{1}{f_0}\right)^{|A_3|}\left(\frac{1}{f_1}\right)^{|A_2|}\left(\frac{1}{f_2}\right)^{|A_1|}\left(\frac{1}{f_3}\right)^{|A_0|},$$

where f_i*,* $(0 \le i \le 3)$ *denotes, as usual, the number of* i*-faces.*

Proof. First we split the small cubes into some sets, after the number of colored faces. We also have to take into account the different lengths that the edges have. Let A_i, $(0 \le i \le 3)$ be the sets

$$A_i = \{\text{the small cubes have } i \text{ painted faces}\}.$$

We have

$$\begin{aligned} |A_3| &= 8, \\ |A_2| &= 4\left[(m-2)+(n-2)+(p-2)\right], \\ |A_1| &= 2\left[(m-2)(n-2)+(n-2)(p-2)+(p-2)(m-2)\right], \\ \text{and} \quad |A_0| &= (m-2)(n-2)(p-2). \end{aligned}$$

Proceeding in the same way as we have done previously, we get

$$P(3;m,n,p) = P(A_3)\cdot P(A_2|A_3)\cdot P(A_1|A_3\cap A_2)\cdot P(A_0|A_3\cap A_2\cap A_1).$$

Carrying out the same procedure as in Theorem 1 we obtain the same final formula

$$P(3;m,n,p) = \frac{|A_3|!\cdot|A_2|!\cdot|A_1|!\cdot|A_0|!}{(mnp)!}\left(\frac{1}{f_0}\right)^{|A_3|}\left(\frac{1}{f_1}\right)^{|A_2|}\left(\frac{1}{f_2}\right)^{|A_1|}\left(\frac{1}{f_3}\right)^{|A_0|},$$

where f_i, $(0 \le i \le 3)$ denote, as usual, the number of i-faces. Notice that the only modification is in the expression of $|A_i|$. □

Now we state and prove a result for a general hyperrectangle.

Theorem 5. *Consider a hyperrectangle of dimension d, with the edges equal to $m_1, \cdots, m_n$, made up of $m_1, \cdots, m_n$ white unit hypercubes of dimension d. If the same treatment as to the hypercube in Theorem 3 is applied, then*

$$|A_i| = 2^i\left[\sum_{1\le j_1<j_2<\cdots<j_{d-i}\le d}(m_{j_1}-2)\ldots(m_{j_{d-i}}-2)\right], \text{ and}$$

$$P(d;m_1,\ldots,m_d) = \frac{\prod_{0\le i\le d}|A_i|!}{(m_1\cdots m_d)!}\prod_{0\le i\le d}\left(\frac{1}{f_{d-i}}\right)^{|A_i|},$$

where $f_i = \binom{d}{i}2^{d-i}$.

Proof. The proof is based on the fact that the structure of this kind of hyperrectangle is similar to the usual one, so the number of i-faces is also f_i.

Using the experience we have gained in the previous proof, we deduce easily that the only modifications appear for the $|A_i|$. These formulas reflect the asymmetry of the considered structure. We have,

$$\begin{aligned}
|A_d| &= 2^d, \\
|A_{d-1}| &= 2^{d-1}\left[\sum_{j=1}^{d}(m_j-2)\right], \\
|A_{d-i}| &= 2^{d-i}\left[\sum_{1\le j_1<j_2<\cdots<j_i\le d}(m_{j_1}-2)\ldots(m_{j_i}-2)\right], \\
|A_0| &= (m_1-2)\ldots(m_d-2).
\end{aligned}$$

Using the same argument as in Theorem 3 we finally obtain

$$P(d;m_1,\ldots,m_d)=\frac{\prod_{0\le i\le d}|A_i|!}{(m_1\ldots m_d)!}\prod_{0\le i\le d}\left(\frac{1}{f_{d-i}}\right)^{|A_i|}.$$

This allows us to end the proof. □

A particular case of the preceding result is the following corollary.

Corollary 1. *Suppose we have a rectangle made of squares of face* 1*, having the sides of length* m *and* n*, respectively. If after painting the faces someone splits the rectangle, the probability that a blind person assembles a rectangle that looks like the original one is*

$$\begin{aligned}
&P(2;m,n)= \\
&\frac{4!\,(mn-2(m+n)+4)!\,(2(m+n)-8)!}{(mn)!}\left(\frac{1}{f_2}\right)^{(m-2)(n-2)}\times \\
&\left(\frac{1}{f_1}\right)^{2(m+n)-8}\left(\frac{1}{f_0}\right)^{4},
\end{aligned}$$

where $f_i=\binom{2}{i}2^{2-i}$, $0\le i\le 2$.

Proof. The probability of the case presented in the preceding corollary can be easily obtained directly putting in the general formula $|A_2|=4$, $|A_1|=2\left[(m-2)+(n-2)\right]$, and $|A_0|=(m-2)(n-2)$. □

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