A LINDELÖF PROPERTY FOR UNIFORMLY NORMAL FAMILIES

RICHARD E. BAYNE AND MYUNG H. KWACK

Abstract. In this note we present a simple new proof for a Lindelöf property for a normal map accessible to advanced undergraduate students. The proof extends the result to uniformly normal families: If $\{f_n : D \to P^1(\mathbb{C})\}$ is a uniformly normal sequence from the unit disk D in the complex plane $\mathbb{C}$ into the Riemann Sphere $P^1(\mathbb{C})$ such that $\lim_{r_n \to 1} f_n(r_n)$ exists for all $\{r_n\} \subset (0, 1)$, then the sequence $\{f_n\}$ has non-tangential limit at 1.

1. Introduction

In 1912, P. Montel proved his namesake theorem below based on normal families [9]. This shed an important light on the remarkable theorem due to P. Fatou in 1906 that guarantees the existence of radial limits for a bounded holomorphic function defined on the unit disk almost everywhere on the boundary [3].

Theorem 1.1. [5] *Let $f : D \to \mathbb{C}$ be a bounded holomorphic function defined on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ into the complex plane $\mathbb{C}$. If f has radial limit at 1, then f has non-tangential limit at 1.*

The Lindelöf Principle, extending this theorem, was proved by Lindelöf in 1915 [8]. A nice proof based on harmonic measure is presented in Carathéodory [3, Vol. II, p. 41].

Theorem 1.2. *(Lindelöf Principle)* [3] *Let $f : D \to \mathbb{C}$ be a bounded holomorphic function defined on the unit disk D. Suppose that J is a Jordan curve in D ending at $1 \in \partial D$ such that $\lim_{J \ni z \to 1} f(z) = c$. Then f has non-tangential limit at 1.*

More details on the history of these theorems can be found in R. Burckel [1, p. 161 and p. 441].

Lehto and Virtanen [7] were led to the concept of normal meromorphic functions defined in a simply connected domain in searching for meromorphic functions with similar boundary behavior and proved the following theorem.

Theorem 1.3. [7] *Suppose $f : \Omega \to P^1(\mathbb{C})$ is a normal meromorphic function from a simply connected hyperbolic domain Ω into the Riemann sphere $P^1(\mathbb{C})$ with asymptotic value p at a boundary point A along a Jordan curve $J \subset \Omega$. Then f has non-tangential limit at A.*

In this note the theorems above, with the additional requirement that the Jordan curve be contained in a nontangential region, are extended to Hayman's uniformly normal families [4] of which the singleton sets of normal functions are members. The proof presented in this paper is simple enough to be accessible to advanced undergraduate students.

The space of holomorphic maps from a Riemann surface Ω to a Riemann surface Y with the compact-open topology will be denoted by $\mathcal{H}(\Omega, Y)$ and $F \circ G = \{f \circ g : f \in F, g \in G\}$ where F and G are a family of maps.

Definition 1. A sequence $\{f_n\} \subset \mathcal{H}(D, Y)$ of maps from the unit disk D to a Riemann surface Y has asymptotic value $p \in Y$ along a Jordan arc $J \subset D$ ending at 1 if $\lim_{J \ni r_n \to 1} f_n(r_n) = p$. If J is the radial line $\overline{01}$, the asymptotic value, $\lim_{J \ni r_n \to 1} f_n(r_n)$, is called radial limit at 1.

A non-tangential region Γ_α for $\alpha > 1$ and an angular region A_θ for $\theta \in (0, \frac{\pi}{2})$ at $1 \in D$ are defined as follows (see figure 1):

$$\Gamma_\alpha = \{z \in D : \frac{|1-z|}{1-|z|} < \alpha\},$$

$$A_\theta = \{z \in D : \pi - \theta < \arg(z-1) < \pi + \theta\}.$$

It is to be noted that non-tangential regions and angular regions are equivalent: For every $\alpha > 1$ there is a $\theta \in (0, \frac{\pi}{2})$ such that $\Gamma_\alpha \subset A_\theta$ and for every $\theta \in (0, \frac{\pi}{2})$ there is an $\alpha > 1$ and a disk Δ centered at 1 such that $A_\theta \cap \Delta \subset \Gamma_\alpha$.

To see this let Δ_1 be the disk that has the line segment $\overline{01}$ as its diameter, $z \in D$ and $\phi = \pi - \arg(z-1)$ (see figure 2). From the law of cosines

$$\frac{|1-z|}{1-|z|} = \frac{(1+|z|)(|1-z|)}{1-|z|^2} = \frac{(1+|z|)}{2\cos\phi - |1-z|}.$$

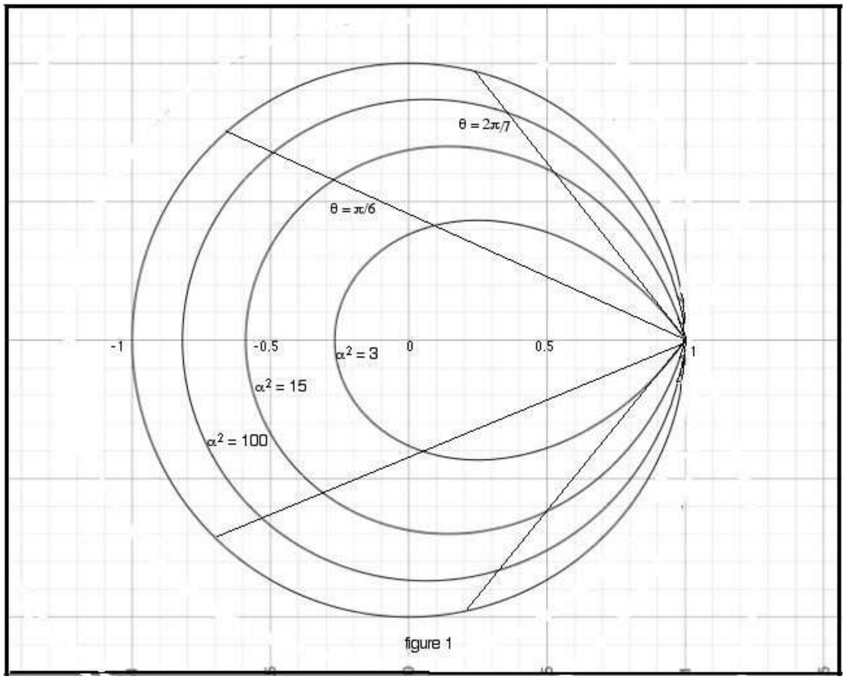

figure 1

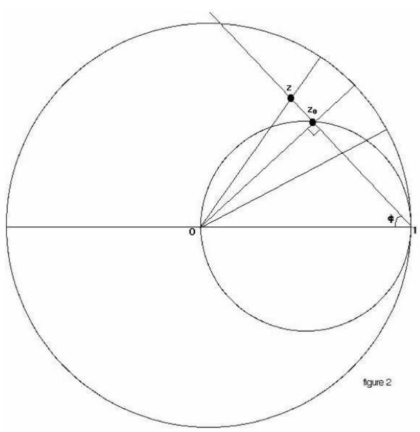

figure 2

Thus,

$$\frac{1}{2\cos\phi} \le \frac{|1-z|}{1-|z|} \quad \text{for} \quad z \in D,$$

and

$$\frac{|1-z|}{1-|z|} \le \frac{2}{\cos\phi} \quad \text{for} \quad z \in D \cap \Delta_1.$$

Definition 2. A sequence $\{f_n\} \subset \mathcal{H}(D,Y)$ from the unit disk D to a Riemann surface Y has non-tangential (or angular) limit at 1 if $\lim_{\Gamma_\alpha \ni z_n \to 1} f_n(z_n)$ exists for all non-tangential regions $\Gamma_\alpha,\ \alpha > 1$, at 1.

2. Main Results

The non-Euclidean distance on the unit disk D and the spherical distance on the Riemann sphere $P^1(\mathbb{C})$ are denoted by ρ and χ, respectively.

Main Theorem 1. *Suppose $\{f_n\} \subset \mathcal{H}(D, P^1(\mathbb{C}))$ is a relatively compact sequence with asymptotic value p at 1 along a Jordan arc J in an angular region A_θ at 1 for some $\theta \in (0, \frac{\pi}{2})$. Then the following are equivalent.*

(1) *The sequence $\{f_n\}$ does not have non-tangential limit at 1.*

(2) *There exist an angular region $A_{\theta'}$ at 1 for some $\theta' \in (0, \frac{\pi}{2})$ and a sequence $\{g_n\} \subset \mathcal{H}(D, A_{\theta'})$ such that the sequence $\{f_n \circ g_n\}$ is not relatively compact.*

(3) *There exist an angular region $A_{\theta'}$ at 1 for some $\theta' \in (0, \frac{\pi}{2})$ and sequences $\{z_n\}$ and $\{w_n\}$ in $A_{\theta'}$ both converging to 1 such that $\lim \rho(z_n, w_n) = 0$ and $\lim \chi(f_n(w_n), f_n(z_n)) \neq 0$.*

Proof. (1) $\Rightarrow$ (2). It may be assumed that for some neighborhood U of p there is a sequence $\{z_n\} \subset A_{\theta'},\quad 0 < \theta < \theta' < \theta'' < \frac{\pi}{2}$, with $|1 - z_n| \to 0$, $\Im z_n \ge 0$ and $f_n(z_n) \notin U$. Let L''_+, L''_-, L'_+, and L'_- be the rays given by

$$L''_\pm = \{z \in D : \arg(z-1) = \pi \pm \theta''\} \text{ and}$$

$$L'_\pm = \{z \in D : \arg(z-1) = \pi \pm \theta'\}.$$

Let r_n be the intersection point with the x-axis of the line through z_n perpendicular to L''_- (see figure 3). Define $g_n : D \to A_{\theta''}$ by

$$g_n(z) = z(1 - r_n)\sin\theta'' + r_n.$$

Suppose a subsequence of the sequence, again denoted by $\{f_n \circ g_n\}$, converges uniformly on compact subsets of D to $\alpha \in \mathcal{H}(D, P^1(\mathbb{C}))$. Let q_n be an intersection point with the curve J of the line through r_n perpendicular to either L''_- or L''_+ (see figure 3). Let $b_n = g_n^{-1}(q_n)$. A subsequence, say $\{b_n\}$, converges to $b \in D$ with

$$|b| \le 1 - \frac{\tan(\theta'' - \theta)}{\tan\theta''} < 1.$$

Let $D_s(b)$ be the disk centered at b with radius $s = \frac{1-|b|}{2}$. Let

$$A = \bigcap_{m \geq 1} \overline{\bigcup_{k \geq m} g_k^{-1}(J \cap g_k(D_s(b)))}.$$

There is a continuum $J' \subset A$ with $b \in J'$ [11, p. 209] such that

$$\text{For all } z \in J', \quad f_n \circ g_n(z) \to p = \alpha(z).$$

Thus, α is constant in $D_s(b)$ and so in D. Let $a_n = g_n^{-1}(z_n)$. Then

$$|a_n| \leq 1 - \frac{\tan(\theta'' - \theta')}{\tan \theta''} < 1.$$

Hence, a subsequence, again denoted by $\{a_n\}$, converges to a with $|a| \in [0, 1)$ and

$$\lim f_n(z_n) = \lim f_n \circ g_n(a_n) = \alpha(a) = p,$$

which is a contradiction. It follows that the sequence $\{f_n \circ g_n\}$ is not relatively compact in a neighborhood of a.

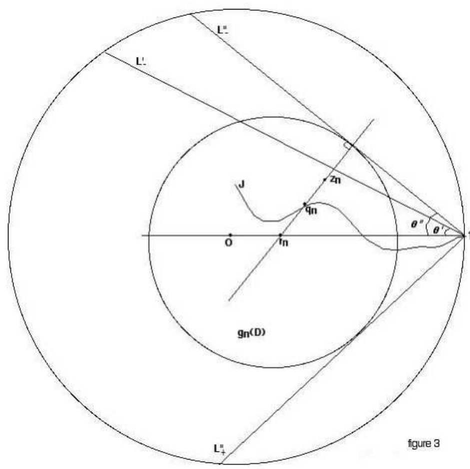

figure 3

(2) $\Rightarrow$ (3). Since the sequence $\{f_n \circ g_n\}$ is not relatively compact, there is a point $a \in D$ and a sequence $\{a_n\}$ converging to a such that

$$\lim f_n \circ g_n(a) \neq \lim f_n \circ g_n(a_n).$$

Relative compactness of the sequence $\{f_n\}$ implies that $\lim g_n(a) = 1$. By the Schwarz-Pick Lemma,

$$\rho(g_n(a_n), g_n(a)) \leq \rho(a_n, a) \to 0.$$

(3) $\Rightarrow$ (1). Obvious. $\square$

Sequences with the property given in Main Theorem 1 (3) were called sequences of 'W-points' for the function f [2]. The following corollaries extend Theorem 1.1 and Theorem 1.2 but with the added requirements that the Jordan curve be contained in an angular region.

Corollary 2.1. *Let $\{f_n\} \subset \mathcal{H}(D,\mathbb{C})$ be a uniformly bounded sequence. If the sequence $\{f_n\}$ has radial limit at* 1*, then it has non-tangential limit at* 1.

Corollary 2.2. *Let $\{f_n\} \subset \mathcal{H}(D,\mathbb{C})$ be a uniformly bounded sequence with an asymptotic value p at* 1 *along a Jordan curve J in an angular region A_θ at* 1 *for some $\theta \in (0, \frac{\pi}{2})$. Then the sequence $\{f_n\}$ has non-tangential limit at* 1.

Lehto and Virtanen made the following definition of normal meromorphic functions. The family of the holomorphic automorphisms of a domain $\Omega \subset \mathbb{C}$ is denoted by $\mathcal{A}(\Omega)$. The term 'a normal family' is frequently used in place of a relatively compact family in $\mathcal{H}(\Omega, P^1(\mathbb{C}))$.

Definition 3. [7] A meromorphic map $f \in \mathcal{H}(\Omega, P^1(\mathbb{C}))$ from a simply connected domain Ω is normal if the family $\{f \circ g : g \in \mathcal{A}(\Omega)\}$ is normal.

A meromorphic map $f \in \mathcal{H}(D, P^1(\mathbb{C}))$ is normal [2, 10] if and only if

$$\sup\{(1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2}, z \in D\} < \infty.$$

Thus, a meromorphic map $f \in \mathcal{H}(D, P^1(\mathbb{C}))$ is normal if and only if there is a constant $c > 0$ such that

$$\text{for all } z,\ w, \chi(f(z), f(w) \leq c\rho(z,w).$$

Since every $g \in \mathcal{H}(D,D)$ satisfies

$$\text{for all } z,\ w,\quad \rho(g(z), g(w) \leq \rho(z,w) \quad \text{(Schwarz-Pick Lemma)},$$

the family $\{f \circ g : g \in \mathcal{H}(D,D)\}$ is, then, equicontinuous and thus relatively compact whenever $f \in \mathcal{H}(D, P^1(\mathbb{C}))$ is a normal map.

A family $F \subset \mathcal{H}(D, \mathbb{C})$ is called invariant if $F \circ \mathcal{A}(D) = F$. Isolating some properties of an invariant normal family, Hayman [4] introduced uniformly normal families.

Definition 4. A family $\mathcal{F} \subset \mathcal{H}(D, \mathbb{C})$ is called *uniformly normal* if there is a $B \geq 0$ such that whenever $f \in \mathcal{F}$, $z_1, z_2 \in D$, $|f(z_1)| \leq 1$ and $|f(z_2)| \geq e^B$, then $\rho(z_1, z_2) \geq 1/2$.

Proposition 2.1. [6] *Let $\mathcal{F} \subset \mathcal{H}(D, \mathbb{C})$ be an invariant family. Then $\mathcal{F}$ is uniformly normal if and only if it is normal.*

These uniformly normal families by Hayman are characterized by the following proposition [6] whose proof is straightforward.

Proposition 2.2. *Let $\mathcal{F} \subset \mathcal{H}(D, \mathbb{C})$. The following are equivalent.*

(1) *$\mathcal{F}$ is uniformly normal.*
(2) *$\mathcal{F} \circ \mathcal{H}(D, D)$ is uniformly normal.*
(3) *$\mathcal{F} \circ \mathcal{H}(D, D)$ is relatively compact in $\mathcal{H}(D, P^1(\mathbb{C}))$.*
(4) *$\mathcal{F} \circ \mathcal{A}(D)$ is normal.*
(5) *$\mathcal{F}$ is a subset of an invariant normal family.*

Remark 2.1. *Besides the invariant normal families, examples of uniformly normal families include families of holomorphic maps into hyperbolic domains including $\mathcal{H}(D, \mathbb{C} - \{0, 1\})$.*

Uniformly normal families are necessarily normal but normal families may not be uniformly normal. Let $f(z) = e^{\frac{i}{1-z}}$. The singleton set $\{f\} \subset \mathcal{H}(D, \mathbb{C})$ is normal but not uniformly normal since the map f is not a normal map: the family $\{f \circ \phi : \phi \in \mathcal{A}(D)\}$ is not normal.

From Proposition 2.2, Hayman's definition of uniformly normal families may be extended to families of maps into the Riemann Sphere. These classes of uniformly normal families include the singleton sets of normal maps defined by Lehto and Virtanen.

Definition 5. [6] Let Ω be a domain in $\mathbb{C}$. A family $\mathcal{F} \subset \mathcal{H}(\Omega, P^1(\mathbb{C}))$ is uniformly normal if $\mathcal{F} \circ \mathcal{H}(D, \Omega)$ is relatively compact in $\mathcal{H}(D, P^1(\mathbb{C}))$.

The following corollary is a 'generalization' of Theorem 1.3 by Lehto and Virtanen [7] of the Lindelöf Principle with the added requirements that the Jordan curve be contained in an angular region.

Corollary 2.3. *Suppose $F \subset \mathcal{H}(D, P^1(\mathbb{C}))$ is a uniformly normal family. If $\{f_n\} \subset F$ is a sequence with asymptotic value at 1 along a Jordan curve $J \subset A_\theta$ for some angular region A_θ, $\theta \in (0, \frac{\pi}{2})$ at 1, then the sequence has non-tangential limit at 1.*

Corollary 2.4. *Suppose* $\{f_n\} \subset \mathcal{H}(D, P^1(\mathbb{C}))$ *is a sequence with asymptotic value at* 1 *along a Jordan curve* $J \subset A_\theta$ *for some angular region* A_θ, $\theta \in (0, \frac{\pi}{2})$ *at* 1. *If the sequence of the restrictions of each* f_n *to any angular region at* 1 *in* D *is uniformly normal, then the sequence* $\{f_n\}$ *has non-tangential limit at* 1.

Remark 2.2. *A theorem of Abel* [3, Vol. I, p. 215] *shows a sequence that has non-tangential limit at* 1. *If a sequence* $f_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ *converges to* $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ *with unity as the radius of convergence and the series* $s = a_0 + a_1 + \cdots$ *converges, then the sequence* $f_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ *has non-tangential limit* s *at* 1. *It was shown that the sequence of restrictions of each* f_n *to any angular regions at* 1 *in* D *is uniformly normal. It also follows that the limit* f *has non-tangential limit* s *at* 1 *as the restriction of* f *to any angular region at* 1 *of* D *is normal.*

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Department of Mathematics, Howard University, Washington DC 20059
E-mail address: rbayne@scs.howard.edu

Department of Mathematics, Howard University, Washington DC 20059
E-mail address: mkwack@howard.edu