ANOTHER SIMPLE CONSTRUCTION OF SMITH NUMBERS

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ABSTRACT. We find that for any prime $P > 5$ with digital sum equals 5, the number $21P$ is Smith, and so is $112P$.

1. A Brief History

For natural numbers n, the *digital sum* $S(n)$ denotes the sum of the digits in n. Moreover, the notation $S_p(n)$ stands for the sum of the digital sums of the prime factors of n, counting multiplicity. We call a composite n a *Smith* number when $S(n) = S_p(n)$. For example, the number 728 is Smith because $S(728) = 7 + 2 + 8 = 17$ and, since $728 = 2 \times 2 \times 2 \times 7 \times 13$, we also have $S_p(728) = 2 + 2 + 2 + 7 + 1 + 3 = 17$.

Smith numbers were named in 1982 by Albert Wilansky [10] after Harold Smith, his brother-in-law whose telephone number had this peculiar property. Just five years after Wilansky's article, Wayne McDaniel [6] gave a constructive proof for the existence of infinitely many Smith numbers. In his proof, McDaniel produced a sequence of Smith numbers of the form $t \times 9R_n \times 10^k$, where $t \in \{2, 3, 4, 5, 7, 8, 15\}$ and R_n is a repunit of length n, i.e., $R_n = (10^n - 1)/9$.

Before McDaniel's proof, Oltikar and Wayland [8] had demonstrated that if the repunit R_n is prime, $n > 2$, then $3304R_n$ is a Smith number. The choice of 3304 was not unique as many other substitutes were later discovered and now listed as Sloane's A104167 [9].

A similar but different infinite sequence involving repunits was later given by Costello and Lewis [3] in 2002, thereby providing an alternate proof for the existence of infinitely many Smith numbers. The sequence under consideration here consists of Smith numbers of the form $11^j \times 9R_n \times 10^k$.

In 1984, Pat Costello [2] also showed how to construct Smith numbers in the form $n = P \times Q \times 10^k$, where P is a small prime and Q is a Mersenne prime. The idea behind this construction is quite simple: If PQ is not already Smith, keep multiplying PQ by 10. Doing so will not alter $S(PQ)$, while it repeatedly adds 7 to $S_p(PQ)$—hopefully until we reach the equality

$S(n) = S_p(n)$, or else we try again with a different P. In fact, the role of a Mersenne prime here is not really needed—unless one aims for a record prime to generate a largest Smith number.

Another construction of Smith numbers was given by Samuel Yates [11] in 1986, in the form of a prime times two. The only condition for such prime P to give rise to a Smith number $n = 2P$ is that $S(2P) - S(P) = 2$, a property which can be efficiently tested by a simple checking on the digits in P.

Yet another result by Yates [12] claimed that if the palindromic number $Q = 10^{2m} + A \times 10^m + 1$ is a prime, then we can find a Smith number of the form $9R_n \times Q^j \times 10^k$, where R_n is also required to be prime. Using this construction, Costello [1] discovered the then-largest Smith number of well over 32 million digits. Earlier, McDaniel [7] had stated that there are infinitely many palindromic Smith numbers, if numbers of the form $Q \times 10^k$ are considered palindromic whenever Q is.

A very readable historical account on Smith numbers up to 1994 can be found in an article by Underwood Dudley [4].

2. A New Construction

Our sole purpose here is to introduce an alternate, very simple construction of Smith numbers relying on primes with digital sums equal to 5. It is not known whether or not there exist infinitely many such primes, although we observe (see Table 1) that such primes seem common.

Theorem 2.1. *Let $P > 5$ be any prime with $S(P) = 5$. Then $21P$ is a Smith number.*

The key facts in proving Theorem 2.1 are the well-known rules $S(10n) = S(n)$ and $S_p(mn) = S_p(m) + S_p(n)$, for natural numbers m and n. Moreover, if there is no carry in the addition $m+n$, then $S(m+n) = S(m)+S(n)$.

Proof. Note that $S_p(21P) = 3+7+5 = 15$. Now one family of primes with digital sums 5 is given by the form $4 \times 10^k + 1$, where $k \geq 1$. In this case, $21P = 84 \times 10^k + 21$ also has digital sum 15 for each k.

Outside this class, such a prime P is composed of the digits 0 up to 3. Thus, $S(20P) = S(2P) = 2 \times 5 = 10$. Moreover, the largest digit appearing in $20P$ is maximum 6, and so adding $20P$ into P will never involve a carry in the process. Hence, $S(21P) = S(20P) + S(P) = 10 + 5 = 15$, proving the claim. □

Up to 10^5, there are fourteen Smith numbers which have the form $21P$, where P is prime with $S(P) = 5$. These are

483, 861, 2373, 2751, 6531, 8421, 21273,
21651, 23163, 27321, 42063, 44331, 63231, 84021,

which correspond to the values of $P = 23$, 41, 113, 131, 311, 401, 1013, 1031, 1103, 1301, 2003, 2111, 3011, and 4001, respectively. This list of primes can be found in Sloane's A062341 [9].

Using Sage software, we provide further examples of Smith numbers generated in this way by searching for primes with digital sums 5 of the form $4 \times 10^k + 1$ and of several other forms, up to $k \leq 1000$. The results are collected in Table 1.

TABLE 1. Examples of primes P with $S(P) = 5$, hence of Smith numbers $n = 21P$, tested up to $k \leq 1000$ in each given form.

P	k
$2 \times 10^k + 3$	1, 3, 5, 6, 7, 12, 16, 17, 22, 24, 35, 115, 120, 358
$4 \times 10^k + 1$	1, 2, 3, 13, 229, 242, 309, 957
$11 \times 10^k + 3$	1, 2, 3, 10, 11, 15, 21, 68, 127, 220
$13 \times 10^k + 1$	1, 2, 3, 7, 16, 53, 95, 105, 125, 163, 358, 423, 562
$22 \times 10^k + 1$	6, 10, 11, 102, 146, 296
$31 \times 10^k + 1$	1, 51, 65, 336, 747
$101 \times 10^k + 3$	1, 2, 4, 44, 55, 67, 359, 391
$103 \times 10^k + 1$	1, 3, 7, 20, 21, 37, 55, 115, 195, 251, 363, 897
$112 \times 10^k + 1$	4, 5, 15, 27, 31, 86, 248, 476, 658, 682
$121 \times 10^k + 1$	3, 134, 168, 215, 264, 602, 827, 971
$202 \times 10^k + 1$	3, 4, 15, 39, 83, 221, 591
$211 \times 10^k + 1$	1, 4, 7, 8, 16, 32, 43, 242, 510, 700
$301 \times 10^k + 1$	1, 4, 7, 39, 51, 87, 160, 285, 543, 565, 705
$1001 \times 10^k + 3$	2, 3, 8, 12, 13, 15, 59, 78, 79, 155, 175, 222, 358, 403, 405, 423
$1003 \times 10^k + 1$	5, 6, 10, 28, 52, 108, 161, 310, 410, 438, 616, 756, 899
$1012 \times 10^k + 1$	36, 37, 40, 56, 78, 153, 171, 200, 276, 406, 502, 517, 750
$1021 \times 10^k + 1$	1, 5, 10, 13, 61, 85, 350, 361
$1102 \times 10^k + 1$	4, 9, 16, 19, 24, 30, 226, 286, 747
$1111 \times 10^k + 1$	7, 14, 49, 357, 437
$1201 \times 10^k + 1$	9, 10, 15, 20, 48, 60, 70, 268, 339, 442, 466
$2002 \times 10^k + 1$	4, 19, 24, 35, 37, 39, 40, 47, 50, 53, 78, 85, 118, 184, 358, 629, 883
$2011 \times 10^k + 1$	5, 6, 8, 11, 28, 40, 86, 784
$2101 \times 10^k + 1$	8, 32, 76, 79, 120, 130, 132, 134, 440
$3001 \times 10^k + 1$	5, 6, 163, 227, 308

Lastly, our larger random example is the Smith number

$$21 \times (3001 \times 10^{9723} + 1),$$

which has 9,728 decimal digits. The prime $3001 \times 10^{9723} + 1$ was found using the Proth.exe program written by Yves Gallot [5], based on Proth's theorem for the primality of numbers of the form $k \times 2^n + 1$. (The program actually employs Pocklington's theorem to deal with the generalized form $k \times b^n + 1$.)

3. A Challenge

With a little bit more checking, we can prove that Theorem 2.1 remains valid when the multiplier 21 is replaced by 112.

Theorem 3.1. *Let $P > 5$ be any prime with $S(P) = 5$. Then $112P$ is a Smith number.*

Proof. This time $S_p(112P) = 2 + 2 + 2 + 2 + 7 + 5 = 20$. Then note that in the addition process to compute $100P + 10P + P$, the position of each digit is simply shifted one to the right from $100P$ to $10P$ and from $10P$ to P. Hence, in each "column" the resulting sum is no larger than three consecutive digits in P—in particular, 5 is maximum. This implies that first, the addition $100P + 10P + P$ involves no carries and second, that adding P to $111P$ is yet carry-free because the largest digit in P is 4 or less. We may now deduce that $S(111P) = 5 + 5 + 5 = 15$ and $S(112P) = 5 + 15 = 20$, as desired. □

We leave as a challenge to the reader to try to find other multipliers m for which the number mP is Smith for every prime P of digital sum $S(P) = 5$ or of a fixed other digital sum, preferably small.

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