

AL-RISĀLA AL-MUHĪTĪYYA: A SUMMARY

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ABSTRACT. In this paper we present an English summary of *al-Risāla al-muhītīyya* (“The Treatise on the Circumference”). This treatise is one of the most significant mathematical achievements of Ghiyāth al-Dīn Jamshīd Mas’ūd al-Kāshī which he completed in July 1424 (Sha’bān 827 A.H.L.). Also, we present a complete English translation of the titles of all ten sections as well as the title of the introduction and the title of the conclusion. Additionally, we provide comments throughout the paper as necessary.

1. PRELIMINARIES

Ghiyāth al-Dīn Jamshīd Mas’ūd al-Kāshī (also known as al-Kāshānī, Kāshī, or Kāshānī) was one of the most renowned mathematicians in the Islamic world. Al-Kāshī’s exact date of birth is not known, but it was sometime during the second half of the fourteenth century in the city of Kāshān in central Iran. Al-Kāshī died on the morning of Wednesday, June 22, 1429 (Ramadān 19, 832 A.H.L.) outside of Samarqand at the observatory he had helped to build in Uzbekistān. For a more detailed discussion of Al-Kāshī’s life and his mathematical achievements we refer the reader to the author’s article in the *Journal of Recreational Mathematics* [5].

One of the most significant mathematical achievements of al-Kāshī is *al-Risāla al-muhītīyya* (“The Treatise on the Circumference”), which is also known as *Risāla-i muhītīyya*, or simply *Muhītīyya*. This 58-page long treatise, written longhand in Arabic by al-Kāshī himself, was completed in Samarqand in July 1424 (Sha’bān 827 A.H.L.) [1]. We argued in [2] that [1] is the original manuscript of *al-Risāla al-muhītīyya* which was donated by Nāder Shāh (King Nāder) in 1145 A.H.S. to the library of the Āstāne Qudse Razawī, Mashhad, Iran. It is Number 162 of the mathematics collection (Number 5389 of the general collection) of the central library of the Āstāne Qudse Razawī. We include photocopies of the first page (Plate A) and the last page (Plate B) of this original manuscript of *al-Risāla al-muhītīyya* with this paper. Another handwritten manuscript of *al-Risāla al-muhītīyya* exists in the Kitābkhān-i Showrā-ye Mellī (National Library of the Parliament), Tehran, Iran. Other manuscripts of this treatise exist

in Germany, India, Russia, Turkey, and various libraries throughout the Middle East and elsewhere.

Paul L. Luckey was the first to write an impressive German translation of most of *al-Risāla al-muhītīyya* in 1949. Luckey's German translation with a detailed commentary, along with copies of most pages from the typed Arabic manuscript number 576 of the Istanbul Military Museum, was published in Berlin, posthumously in 1953 [10]. The typed pages of the Arabic manuscript of *al-Risāla al-muhītīyya* that were used by Luckey and were published along with his translation contain some typographical errors [3]. A. Qurbānī in *Kāshānī Nāmeḥ* [12], a monograph on al-Kāshī and his achievements in mathematics and astronomy, has written a Persian summary of *al-Risāla al-muhītīyya*, based on Āstāne Qudse Razawī's manuscript, Luckey's German translation, Rosenfeld and Youschkevitch's Russian translation, as well as other sources. Qurbānī's book will be our main source for the writing of this article. In addition, Boris A. Rosenfeld and Adolf P. Youschkevitch wrote a Russian account of *al-Risāla al-muhītīyya* with commentaries in 1954 [18]. According to A. Qurbānī and Professor B. A. Rosenfeld himself, the handwritten manuscript that was available to Rosenfeld and Youschkevitch contained some errors, which carried over into the Russian translation. However, the corrected version of [17] along with extended commentaries was published in 1956 in a book by B. A. Rosenfeld [16]. According to Professor Rosenfeld, [16] contains an exposition by the Turkish astronomer Mīrim Chelebī (d. 1525). Chelebī was the grandson of both the Turkish astronomer and mathematician Salāh al-Dīn Mūsā Qādī zāde al-Rūmī (1360-1437), and Alā al-Dīn Alī al-Qūshjī (d. 1474), another Turkish astronomer who was also a mathematician, philosopher, and linguist.

In this paper we present an English summary of *al-Risāla al-muhītīyya*. Also, we present a complete English translation of the titles of all ten sections as well as the title of the introduction and the title of the conclusion. Additionally, we provide comments throughout the paper as necessary. In this summary we present those topics and ideas of *al-Risāla al-muhītīyya* which we believe are mathematically significant, and easy to communicate in modern notations without getting into unnecessary, insignificant, lengthy discussions, and repeated calculations.

As we will see in the introduction of *al-Risāla al-muhītīyya*, the calculations of π by Archimedes (c. 287-212 BC), Abu'l-Wafā' al-Būzjānī (940-998), and Abū Rayḥān al-Bīrūnī (973-after 1050) motivated al-Kāshī to write this treatise on improving the estimation of π by these three renowned mathematicians. Another motivation for al-Kāshī to write this treatise was the fact that there was a real demand for more precise trigonometric tables in connection with advanced research in astronomy. Al-Kāshī used an

inscribed regular polygon and a circumscribed regular polygon each with 3×2^{28} sides to calculate the ratio of the circumference to its diameter of the corresponding circle to obtain an approximation for π that far surpassed all approximation of π by all previous mathematicians and astronomers throughout the globe at that time. In the review of [10], E. S. Kennedy stated that *al-Risāla al-muhītīyya* was a landmark of Islamic computational technique [MR0055264 (14, 1051a)]. Al-Kāshī's method of obtaining this approximation of π was very similar to that of Archimedes. Paul Luckey in the introduction of [10] states that *al-Risāla al-muhītīyya* is a masterpiece of sexagesimal calculations. However, to make his work accessible to non-astronomers al-Kāshī also performed his calculations in the decimal system and hence made use of decimal fractions. Contrary to popular belief by some scientists and mathematicians, especially in some Persian circles, he may not be the first who discovered decimal fractions. However, he did discover decimal fractions on his own and for his own use without knowledge or use of others' works¹ (the footnotes are listed in Section 14).

Comment 1.1. *The author in [5, p. 36] stated that Luckey translated the entire al-Risāla al-muhītīyya into German and in [3, p. 507] he again wrote that Luckey's translation was a comprehensive one. After studying the German manuscript more closely, the author regrets to admit that Luckey did not translate the entire al-Risāla al-muhītīyya. For example, al-Kāshī started al-Risāla al-muhītīyya with the usual phrase, "In the Name of God, Most Gracious, Most Merciful," but the Arabic copy of al-Risāla al-muhītīyya that Luckey used did not have this phrase and hence, it was not in the German translation. An example of a major omission in the German translation was in Section 4, where out of 28 tables (pages) Luckey has translated only four tables; namely, Tables 1, 2, 15, and 28. However, since all 28 tables were about the root extraction perhaps he did not feel that there was a need to translate all 28 tables. Luckey included all pages from the Arabic manuscript that he translated with one exception. He had a translation of Section 5, but Section 5 was missing from the Arabic pages that he included with the German translation.*

Comment 1.2. *Al-Kāshī used the following units of measurements in al-Risāla al-muhītīyya:*

1 farsakh \approx 12000 adru' \approx 6 km

1 dirā' \approx 24 qasabāt \approx 50 cm

1 isba = 1 finger \approx 6 times the width of a medium-size grain of barley

Width of a medium-size grain of barley \approx 6 sha'rāt (plural of sha'ra)

1 sha'ra = The breadth of a horse (mane) hair

Aдру' is the plural of dirā' \approx 50 cm and it is different from the Iranian zar' \approx 104 cm. We note that the value of dirā' varies from country to

country. For example, in Syria it is about 68 cm while in Egypt it is 58 cm.

Qasabāt is the plural of *qasaba* ≈ 3.55 m. In old Persian it used to be called *qasab* (or *nāb*). However, this measurement is no longer in use in modern day Iran.

Farsakh ≈ 6 km and its plural is *farāsikh* in Arabic and *farsakhha* in Persian. In Persian *farsakh* is also called *farsang* or *parasang*, and in Arabic *farsakh* is also called *farsak*.

Comment 1.3. In both the Arabic and the Persian languages there is not a word which is equivalent to the word “circumference”. In both of these languages we use the word *muhīt* for both the circumference of a circle as well as the perimeter of a geometric figure in the plane. Thus, we see where the word *al-muhītīyya* (“On the circumference”) comes from. However some Persian scientists, including *al-Bīrūnī*, used the word “*dowr*” which means “around” instead of *muhīt* for the circumference.

2. IN THE NAME OF GOD, MOST GRACIOUS, MOST MERCIFUL

Al-Kāshī started *al-Risāla al-muhītīyya* with the usual Islamic praise of the Almighty: “In the Name of God, Most Gracious, Most Merciful,” a common phrase used by devout Muslims before undertaking any task. Al-Kāshī discussed the calculations of π by Archimedes, Abu'l-Wafā' al-Būzjānī, and Abū Rayhān al-Bīrūnī. A complete English translation of the introduction of *al-Risāla al-muhītīyya* with commentaries was presented by the author in [2]. What follows is a summary of [2].

Al-Kāshī begins by writing, “Thanks be to the deserving God Who is aware of the ratio of the diameter to the circumference . . .” and he asked for forgiveness and guidance from the Exalted God. Then, he stated that Archimedes has proven that the circumference of a circle is smaller than three times its diameter by less than $\frac{1}{7}$ of the diameter and larger than three times of the diameter by less than $\frac{10}{71}$ of the diameter. Hence, the difference of these two quantities is $\frac{1}{497}$ of the diameter². Archimedes calculated the perimeter of the inscribed (circumscribed) regular polygons with 96 sides which is less (more) than the circumference of the circle.

Al-Būzjānī obtained an approximation for the half of the chord of one degree arc with the assumption that the diameter of the circle is 120 units. He multiplied this result by 720 to obtain the perimeter of the inscribed regular polygon with 720 sides. He then calculated the perimeter of the similar circumscribed regular polygon with 720 sides as well. He stated that the circumference of this circle would be 376 units and a fraction³, where this fraction is more than 0; 59, 10, 59 and less than 0; 59, 23, 54, 12 and the difference of these two quantities is 0; 0, 12, 55, 12. However, Abū

Rayhān al-Bīrūnī calculated the chord of a two degree arc and he used it to calculate the perimeter of the inscribed regular polygon with 180 sides equal to 6; 16, 59, 10, 48, 0 and the perimeter of the similar circumscribed regular polygon with 180 sides equal to 6; 17, 1, 58, 19, 6. Then, he has taken the average of these two perimeters as the circumference of the circle.

Al-Kāshī concluded the introduction of *al-Risāla al-muhītīyya* by stating that the calculations of π by Archimedes, al-Būzjānī, and al-Bīrūnī were confusing. Thus, he wanted to determine the circumference of a circle with a known diameter in such a way that he would be certain that in a circle in which its diameter is six hundred thousand times the diameter of the Earth, the difference between the result of his calculations and the actual value of the circumference would not reach the breadth of a hair⁴: the breadth of which is one sixth of the width of an average grain of barley. In the last sentence of the introduction al-Kāshī wrote:

... while I am asking for help from the dear bestower God
Who is the guidance to the right path, I wrote this treatise on the determination of the circumference consisting of ten sections and a conclusion, and I named [entitled] it *al-Muhītīyya*.

3. THE FIRST SECTION

ON DETERMINATION OF THE CHORD OF AN ARC THAT IS THE SUM OF AN ARC OF KNOWN CHORD AND HALF OF ITS SUPPLEMENT TO THE HALF CIRCLE

In [3] we provided a complete English translation of the First Section of *al-Risāla al-muhītīyya* along with comments on German, Persian, and Russian translations. Also, we presented an alternative proof of al-Kāshī's fundamental theorem. However, like all other sections, we present a summary of this section from [3] here. In this section of *al-Risāla al-muhītīyya*, al-Kāshī determined the chord of an arc which is the sum of an arc of a known chord and half of its supplement to the half circle. Although this was the foundation and the main ingredient of all of his calculations in this treatise, he did not state it as a theorem. He stated and proved the following seminal theorem which we call al-Kāshī's fundamental theorem [3, p. 501]⁵. Throughout the paper we use \overline{AB} to represent the line segment connecting the points A and B .

Theorem 3.1 (Al-Kāshī's Fundamental Theorem). *If on a semicircle with the diameter $\overline{AB} = 2r$ and the center E we consider an arbitrary arc AG , and we call the middle of the arc GB the point D , and we draw the chord \overline{AD} , then, $r(2r + \overline{AG}) = \overline{AD}^2$.*

Al-Kāshī's proof of his fundamental theorem was based on four different propositions from *Euclid's Elements* [3]. However, Al-Kāshī did not burden the reader with mentioning Euclid or the *Elements* in his proof. This is because *Euclid's Elements* had been well-known and widely used for geometrical proofs at that time. Al-Kāshī's fundamental theorem gave him a distinct advantage over his predecessors by having found the most accurate value of π at that time.

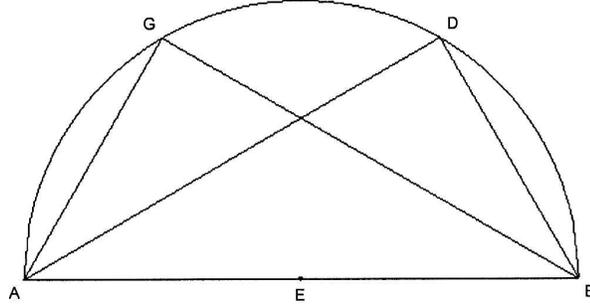


Figure 1

He continued his discussion and he deduced that in a unit circle (Figure 1) if the arc AG is equal to 2θ radians, then, from the right triangle AGB he would get $\overline{AG} = 2 \sin \theta$. Similarly, from the right triangle ADB he obtained

$$\overline{AD} = 2 \sin \frac{1}{2} \left(2\theta + \frac{\pi - 2\theta}{2} \right) = 2 \sin \left(\frac{\pi}{4} + \frac{\theta}{2} \right).$$

Next, he substituted these values of \overline{AG} and \overline{AD} as well as $r = 1$ in his fundamental theorem to obtain

$$2 + 2 \sin \theta = 4 \sin^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right).$$

That is,

$$\sin \left(\frac{\pi}{4} + \frac{\theta}{2} \right) = \sqrt{\frac{1 + \sin \theta}{2}}, \quad \left(\text{where } \theta < \frac{\pi}{2} \right).$$

This is the trigonometric form of Al-Kāshī's fundamental theorem, which is now known as the Lambert identity. It is intriguing to note that this identity appeared in Johann Heinrich Lambert's work in 1770, while al-Kāshī proved it before 1424 (827 A.H.L.), some 346 years earlier than Lambert.

It takes a brilliant mind to come up with such a simple concept and yet with remarkable applications.

In the alternative proof of al-Kāshī's fundamental theorem we used modern notations and applied the Pythagorean theorem to right triangles.

4. THE SECOND SECTION

ON DETERMINATION OF THE PERIMETER OF AN ARBITRARY POLYGON INSCRIBED IN A CIRCLE AND OF THE PERIMETER OF THE SIMILAR POLYGON CIRCUMSCRIBED TO THE SAME CIRCLE

In the Second Section of *al-Risāla al-muhītīyya*, al-Kāshī considered a circle with the diameter $\overline{AB} = 2r$ and he chose the arc \overline{AG} equal to 60° (Figure 2). He called the middle of the arc \overline{BG} the point D , the middle of the arc \overline{BD} the point Z , and the middle of the arc \overline{BZ} the point H . He stated that with the help of his fundamental theorem one can calculate the chord \overline{AD} from the chord \overline{AG} , the chord \overline{AZ} from the chord \overline{AD} , the chord \overline{AH} from the chord \overline{AZ} , and one can continue this process as needed.

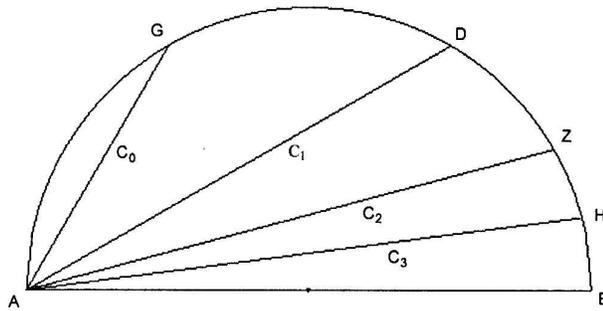


Figure 2

In fact with modern notations and terminologies al-Kāshī has stated a procedure for calculating the chords corresponding to each of the following arcs:

$$\begin{aligned}
 \text{arc } AG &= \alpha_0 = 60^\circ \\
 \text{arc } AD &= \alpha_1 = 180^\circ - 60^\circ = 120^\circ \\
 \text{arc } AZ &= \alpha_2 = 180^\circ - \frac{60^\circ}{2} = 150^\circ \\
 \text{arc } AH &= \alpha_3 = 180^\circ - \frac{60^\circ}{2^2} = 165^\circ \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 \dots \alpha_n &= 180^\circ - \frac{60^\circ}{2^{n-1}}.
 \end{aligned} \tag{4.1}$$

If we call the chord corresponding to the arc α_n by c_n ($n \geq 0$), then the calculation of each chord c_n can be obtained recursively from its preceding chord c_{n-1} by applying al-Kāshī's fundamental theorem. That is,

$$c_n^2 = r(2r + c_{n-1}),$$

and hence,

$$c_n = \sqrt{r(2r + c_{n-1})}. \tag{4.2}$$

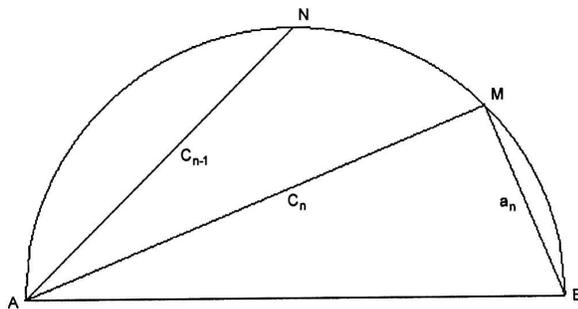


Figure 3

Next, in Figure 3, we let c_n represent the chord \overline{AM} and we let a_n represent the chord \overline{BM} . Now, c_n can be obtained from the above formula and since the triangle AMB is a right triangle we have

$$a_n = \sqrt{4r^2 - c_n^2}.$$

But, a_n is exactly the length of the side of the inscribed regular polygon with 3×2^n sides. Hence, the arc MB can be obtained from the arc AM and Equation (4.1) as

$$\text{arc } MB = 180^\circ - \text{arc } AM = 180^\circ - \left(180^\circ - \frac{60^\circ}{2^{n-1}}\right) = \frac{60^\circ}{2^{n-1}} = \frac{360^\circ}{3 \times 2^n}.$$

Thus, the arc MB is equal to $\frac{1}{3 \times 2^n}$ of the circumference and the chord \overline{MB} is equal to the length of the side of the inscribed regular polygon with 3×2^n sides. Therefore, for any positive integer n one can calculate a_n , the length of each side of the inscribed regular polygon with 3×2^n sides where $a_0 = \sqrt{3}$. For the calculation of the length of each side of the circumscribed regular polygon with 3×2^n sides, al-Kāshī used a circle centered at the point O and radius r and he proceeded as follows (Figure 4).

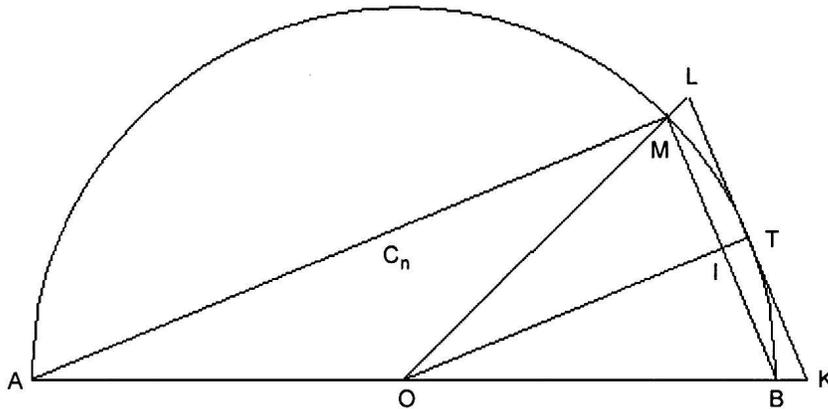


Figure 4

Let the chord \overline{BM} be the length of each side of the inscribed regular polygon with 3×2^n sides. If we let the middle of the arc BM be the point T and we draw \overline{OT} so that it intersects \overline{BM} at the point I and we draw a tangent to the circle at the point T so that it intersects the continuation of \overline{OM} at the point L and it intersects the continuation of \overline{OB} at the point K , then \overline{KL} will be the side of the circumscribed regular polygon with 3×2^n sides which will be similar to the inscribed regular polygon with 3×2^n

sides. Al-Kāshī has proven that⁶

$$\frac{\overline{OI}}{\overline{OT} - \overline{OI}} = \frac{\overline{BM}}{\overline{KL} - \overline{BM}} \quad (4.3)$$

and he has said that since $\overline{OI} = \frac{1}{2}\overline{MA}$ and $\overline{OT} = r$, if \overline{AM} and \overline{BM} are known, then one can calculate \overline{KL} from the Equation (4.3). Therefore, for any non-negative integer n one can calculate \overline{KL} , the length of each side of the circumscribed regular polygon with 3×2^n sides.

5. THE THIRD SECTION

ON HOW MANY SIDES [EQUAL ARCS] WE SUBDIVIDE THE CIRCUMFERENCE AND UP TO WHAT SEXAGESIMAL DIGIT WE SHOULD CALCULATE SO THAT THE CIRCUMFERENCE OF A GIVEN CIRCLE IS OBTAINED WITH AN ACCURACY NOT LARGER THAN THE BREADTH OF A [HORSE] HAIR

In the Third Section of *al-Risāla al-muhītīyya*, al-Kāshī's goal was to determine the number of sides of a circumscribed regular polygon and to use precision in the calculation of its perimeter in a such a way that, as he mentioned in the introduction of *al-Risāla al-muhītīyya*, in a circle where the diameter is six hundred thousand times the diameter of the Earth the difference between the perimeter of the regular polygon and the circumference of the circle does not reach the breadth of a horse hair. Al-Kāshī has said that in a circle where the diameter is six hundred thousand times the diameter of the Earth, its circumference is also six hundred thousand times the circumference of the Earth (greatest circle of the Earth). And with the assumption that the circumference of the Earth is 8000 *farāsikh*⁷, the circumference of the mentioned circle will be 600000×8000 *farāsikh*. He multiplied this value by 12000 so that the circumference of the circle will be in terms of *adru*'. Then, he multiplied the result by 24, 6, and 6 to obtain the measurement of the circumference in terms of *isba*, the width of a medium-size grain of barley, and the width of a horse hair, respectively. After he obtained the circumference of the circle in terms of a horse hair, he has taken $\frac{1}{360}$ of this circumference, namely the arc of one degree, and he has shown that $\frac{1}{60^8}$ of this arc is almost equal to $\frac{4}{5}$ of the width of a hair as follows

$$\frac{600000 \times 8000 \times 12000 \times 24 \times 6 \times 6}{360 \times 60^8} = \frac{200}{243} \approx \frac{4}{5}.$$

He concluded that if he calculates the perimeters of an inscribed and a circumscribed regular polygon in a such a way that the difference of the two perimeters of these two polygons does not reach $\frac{1}{60^8}$, then he would have accomplished his goal.

After applying very clever and skillful computational techniques for approximating the perimeters of the inscribed and the circumscribed regular polygons, he reached the conclusion that in a circle with a radius of 60 units, he must inscribe a regular polygon such that the length of each side of this regular polygon will not be more than $\frac{8}{60^4}$ units. Then, on the right hand side of a table he halved 120° twenty-eight times in sexagesimal system to get

$$120^\circ, 60^\circ, 30^\circ, 15^\circ, \dots, (0; 0, 0, 0, 5, 47, 36, 51, 26, 16, 45, 28, 7, 30)^\circ.$$

On the left hand side of this table he started with an equilateral triangle and he doubled the number of sides of the polygon twenty-eight times in sexagesimal system to obtain

$$1, 2, 8, 16, 12, 48$$

which is equal to $3 \times 2^{28} = 805306368$ in the decimal system. Finally, he concluded that in order for him to achieve his goal he must calculate the length of a side of a regular polygon with 3×2^{28} sides⁸ inscribed in a circle whose radius is 60 units. He has argued in detail that in order for the results of his calculations to be sufficiently precise he must continue his calculations until the order $\frac{1}{60^{18}}$ ⁹ in the sexagesimal system.

6. THE FOURTH SECTION ON THE CALCULATIONS

Section Four is the longest section of *al-Risāla al-muhītīyya*. In fact it is longer than half of the entire *risāla* and it contains more calculations than all other sections of *al-Risāla al-muhītīyya* combined. Al-Kāshī used the formula

$$c_n = \sqrt{r(2r + c_{n-1})}$$

to calculate successively the chords $c_1, c_2, c_3, \dots, c_n$, corresponding to the arcs $120^\circ, 150^\circ, 165^\circ, \dots$, and $180^\circ - \frac{360^\circ}{3 \times 2^n}$, respectively. We note that the chord of 60° , namely c_0 is equal to $1 (= r)$. This section contains 28 *'amāl* (calculations or works)¹⁰, where in each one of these calculations al-Kāshī used root extraction to determine each c_n , for $n = 1, 2, 3, \dots, 28$. He organized the calculation of each c_n in an intensive table, where each table took an entire page of *al-muhītīyya*. He has stated at the start of this section that in each calculation he did not go to the next step until he was completely certain that his calculations in each step were correct. And at the conclusion of the extraction of each root he has squared his result of each extraction of a root and added it to the remainder to make sure that his calculations were completely accurate. He calculated c_{28} with 18

sexagesimal digits after the decimal point as

$$c_{28} = 1, 59; 59, 59, 59, 59, 59, 59, 59, 59, 59, 50, 47, 52, 12, 30, 48, 37, 49, 54, 40.$$

In a unit circle al-Kāshī used the above formula to obtain $c_1, c_2, c_3, c_4, \dots$, and c_{28} in the decimal system as

$$\begin{aligned} c_1 &= \sqrt{3} \\ c_2 &= \sqrt{2 + \sqrt{3}} \\ c_3 &= \sqrt{2 + \sqrt{2 + \sqrt{3}}} \\ c_4 &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}} \\ &\cdot \\ &\cdot \\ &\cdot \\ c_{28} &= \sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{3}}}}, \end{aligned}$$

where there are 28 nested radicals in the calculation of c_{28} . In general, we note that if $r \neq 1$, then $c_0 = r$, and consequently, c_n could be written with n nested radicals as

$$c_n = r \sqrt{2 + \sqrt{2 + \dots + \sqrt{2 + \sqrt{3}}}}.$$

7. THE FIFTH SECTION

ON DETERMINATION OF THE LENGTH OF THE SIDE OF A REGULAR POLYGON INSCRIBED IN A CIRCLE, WHERE THE NUMBER OF SIDES IS 1, 2, 8, 16, 12, 48 [IN SEXAGESIMAL SYSTEM]

The Fifth Section of *al-Risāla al-muhītīyya* is the shortest section. It is only one page long. Actually the entire section is just a single table consisting of the calculation of a_{28} . With the help of the equation

$$a_n = \sqrt{4r^2 - c_n^2}$$

al-Kāshī calculated the length of each side of an inscribed regular polygon in a unit circle. We note that in a unit circle $c_0 = r = 1$, and hence, $a_0 = \sqrt{3}$. His main objective in this section was to calculate in a unit circle

the length of each side of the inscribed regular polygon with 3×2^{28} sides up to 14 digits after the decimal point in sexagesimal system, to obtain

$$a_{28} = 0; 0, 0, 0, 6, 4, 1, 14, 59, 36, 14, 33, 36, 19, 25.$$

Also, al-Kāshī calculated a_{28} in the decimal system using a unit circle to achieve

$$a_{28} = \sqrt{2 - \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}$$

where as in the calculation of c_{28} there are 28 nested radicals in the calculation of a_{28} . In general, we note that if $r \neq 1$, then $c_0 = r$, $a_0 = r\sqrt{3}$, and consequently, a_n could be presented with n nested radicals as

$$a_n = r\sqrt{2 - \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}$$

8. THE SIXTH SECTION

ON DETERMINATION OF THE PERIMETER OF A [REGULAR] POLYGON
 INSCRIBED IN A CIRCLE AND ITS SIMILAR [REGULAR] POLYGON
 CIRCUMSCRIBED TO THE [SAME] CIRCLE, WHERE EACH POLYGON HAS
 805306368 SIDES¹¹

In the Sixth Section of *al-Risāla al-muhītīyya*, al-Kāshī has multiplied the value of a_{28} that he calculated in Section Five by 3×2^{28} to obtain the perimeter of the inscribed regular polygon with 3×2^{28} sides. Similarly, he used the Equation (4.3) to calculate \overline{KL} (Figure 3), the length of one side of the circumscribed regular polygon to obtain the perimeter of the circumscribed regular polygon with 3×2^{28} sides. He has taken the average of these two values as the circumference of the circle which in the sexagesimal system is

$$6, 16; 59, 28, 1, 34, 51, 46, 14, 50.$$

Then, he arranged (versified) this value of the circumference in sexagesimal system into a poem. Out of real concern that scribes might err in the recording of these numbers and thus waste the fruit of his labor, and since he also wanted the multiples of the circumference to be readily available for use, he has multiplied the circumference by integers from 1 to 60. He has recorded these multiples in a table in such a way that if one wants, for example 45 times the circumference, one could immediately obtain it from this table.

Al-Kāshī noted that if one takes the radius of the circle as the unit (instead of 60), the value of the circumference still will be the same, but one has to reduce all of its *marātīb* once¹². This amounts to shifting the decimal point one digit to the left. Hence, according to al-Kāshī the value of 2π in sexagesimal system would be

$$2\pi = 6; 16, 59, 28, 1, 34, 51, 46, 14, 50,$$

where all 9 digits after the decimal point are *daqiq* (exact)¹³. We note that this value of 2π surpassed the practical needs of astronomy at that time: a remarkable achievement!

9. THE SEVENTH SECTION

ON IGNORING [ROUNDING-OFF] THE LAST DIGITS OF THE PRECEDING CALCULATIONS IN ADDITIVE AND SUBTRACTIVE FRACTIONS¹⁴

In the Seventh Section of *al-Risāla al-muhītīyya*, again al-Kāshī discussed in detail that ignoring (rounding off) the last digits of sexagesimal fractions in calculating the value of 2π will not cause any error in the final result of the calculation. For emphasis, at the conclusion of this section yet again, he wrote, that whatever he ignored in the last digits of the sexagesimal fractions in the calculation of 2π , its value will not reach $\frac{1}{60^9}$ in the calculation of the circumference¹⁵.

10. THE EIGHTH SECTION

ON CONVERSION OF THE SIZE OF THE CIRCUMFERENCE INTO INDIAN [ARABIC] NUMERALS ASSUMING THAT THE HALF OF THE DIAMETER IS ONE

In the Eighth Section of *al-Risāla al-muhītīyya*, al-Kāshī calculated the circumference of a unit circle, namely the value of 2π in the decimal system. He used decimal fractions very effectively and obtained the value of 2π as¹⁶

$$2\pi = 6.2831853071795865$$

where all 16 digits¹⁷ after the decimal point are *daqiq* (exact). This indicates al-Kāshī's extreme care in calculating the value of 2π . Again, as in the case of the sexagesimal system, to protect the accuracy of these digits from the negligence of scribes and for the reader to be able to readily use multiples of 2π , al-Kāshī multiplied 2π by 1 through 10. Then, he recorded the 10 resulting numbers in a table in such a way that if one wants the value of, for example 9π , one can obtain it instantly from this table. Al-Kāshī was well aware of the importance of his work in calculating the value of 2π . Thus, in order to preserve the results of his calculations, he arranged (versified) the digits of 2π from left to right in a Persian *bait* (couplet) as well as an Arabic couplet.

11. THE NINTH SECTION

ON THE MANNERS OF CALCULATIONS WITH THE TWO TABLES

In the Ninth Section of *al-Risāla al-muhītīyya*, al-Kāshī explained the way to use the two tables that he obtained, in the seventh and eighth sections, consisting of multiples of 2π in the sexagesimal system as we do in the decimal system. He has shown how to calculate the circumference of a circle if its diameter is known, and the way to calculate the diameter provided the circumference is given. To clarify his explanation further he presented examples for the reader. One example is finding the circumference of a circle whose diameter is equal to $650844\frac{1}{8}$ *Adu'* or *Farāsikh*, and the other example is finding the diameter of a circle whose circumference is $650844\frac{1}{8}$ *Adu'*.

12. THE TENTH SECTION

ON DETERMINATION OF THE DIFFERENCE OF WHAT IS IN COMMON
USE AND FAMOUS AMONG PEOPLE AND WHAT WE HAVE OBTAINED

In the Tenth Section of *al-Risāla al-muhītīyya*, al-Kāshī compared the value that he himself obtained for π with those values that were obtained by his predecessors. He has said that usually the value of π is taken as $3\frac{1}{7}$ and he has calculated the difference between his calculation of π and $3\frac{1}{7}$. He has shown that if the circumference of a circle with the radius 3600 *Adu'* is calculated with his value of π and $3\frac{1}{7}$, the difference between the two circumferences will reach $9\frac{1}{10}$ *Adu'*. Also, he has shown that if the radius of a circle is $7073\frac{1}{8}$ times the diameter of the Earth, then the difference between the two circumferences obtained by using his value of π and $3\frac{1}{7}$ would be more than 177 times the diameter of the Earth. In effect, he has shown that the result of his calculations is much closer to the real value of π than $3\frac{1}{7}$.

13. THE CONCLUSION [OF AL-RISĀLA AL-MUHĪTĪYYA]

ON PROOF OF THE ERRORS OF ABU'L-WAFĀ' [AL-BŪZJĀNĪ] AND ABŪ
RAYHĀN [AL-BĪRŪNĪ]

In the *al-Khātima* (conclusion) of *al-Risāla al-muhītīyya*, al-Kāshī continued his discussion concerning the errors of Abu'l-Wafā' al-Būzjānī and Abū Rayhān al-Bīrūnī that he started in the introduction of *al-Risāla al-Muhītīyya*. However, this time he presented proofs of how both al-Būzjānī and al-Bīrūnī erred in the calculation of π . At the beginning of the conclusion of *al-Risāla al-muhītīyya*, al-Kāshī applied theorems from Book I of *Almagest* to calculate the chord of a one degree arc and the chord of a thirty minute arc. Then he compared the value that he himself obtained for the chord of the thirty minute arc to the value which he attributed to

al-Būzjānī for the chord of a thirty minute arc, and he has shown that the result of his calculation was more accurate (see Comment 13.1). Finally, in the second part of the conclusion of *al-Risāla al-muhītīyya*, al-Kāshī has calculated the chord of a two degree arc. This time he had compared his result with the value that al-Bīrūnī had obtained for the chord of a two degree arc. Again, as expected he has shown that his calculation of the chord of a two degree arc was far more accurate than al-Bīrūnī's incorrect value (see Comment 13.2).

Comment 13.1. *Al-Kāshī did not mention the work of al-Būzjānī that al-Kāshī used as his source for pointing out al-Būzjānī's error. According to Qurbānī [12, p. 137], al-Kāshī most likely was referring to statements made by Nasīr al-Dīn al-Tūsī (1201-1274) in his Taksīr al-Dā'irā which is an Arabic translation of Archimedes' Division of the Circle. In the commentary al-Tūsī stated:*

... For example, let us assume that [the point] C is the center of a circle and the arc AB is equal to $\frac{1}{720}$ of the circumference of that circle. We connect [draw] the chord \overline{AB} . Based on the mentioned principles¹⁸ and al-Būzjānī's calculations, the length of the chord \overline{AB} is equal to 0; 31, 24, 55, 54, 55. Now, if we take the diameter as 120 parts, then this value would be the chord of half of a degree [arc]. Next, if we take this chord as one side of the inscribed [regular] polygon with 720 sides, then the perimeter of this polygon would be 376; 59, 10, 59, ...

This is exactly what al-Kāshī attributed to al-Būzjānī in the introduction of al-Risāla al-Muhītīyya. However, Woepcke [21] has shown that the number that al-Tūsī has taken as the chord \overline{AB} is in fact the sine of one half of one degree and not the chord of one half of one degree which al-Būzjānī himself obtained in his Kitāb al-Kāmīl ("Complete Book"), which is a simplified version of Ptolemy's Almagest. Therefore, it was al-Kāshī who erred and not al-Būzjānī.

Comment 13.2. *According to Qurbānī [12, p. 138], whatever al-Kāshī stated in the introduction of al-Risāla al-muhītīyya about al-Bīrūnī and his calculation of the circumference of a circle was obtained from al-Bīrūnī's Qānoūn al-Mas'ūdī. In Chapter Five of Book III of Qānoūn al-Mas'ūdī, al-Bīrūnī recorded the chord of a two degree arc as 0; 2, 5, 39, 43, 36 which as al-Kāshī pointed out in the introduction of al-Risāla al-muhītīyya is incorrect and is larger than the actual value of the chord of a two degree arc. This larger value of the chord of a two degree arc which al-Bīrūnī used as a side of a regular inscribed polygon with 180 sides resulted in a larger perimeter for this regular inscribed polygon. Also, since he used this value of the chord*

of a two degree arc to obtain the length of one side of a similar regular circumscribed polygon with 180 sides, again he obtained a larger value for the perimeter of the circumscribed polygon than its actual value. He used the perimeters of these inscribed and circumscribed regular polygons in a unit circle to find two approximations of π in the sexagesimal system: namely, $3; 8, 29, 35, 24$ and $3; 8, 30, 59, 10$, respectively. In the decimal system these values are equal to 3.141742 and 3.141744 where both of these values are larger than 3.14166, the known value of π by Greeks and Indians at that time.

14. FOOTNOTES

¹In [3] the author was deceived by claims in some literature that the decimal fractions were discovered by al-Kāshī and he too erroneously stated that al-Kāshī was the discoverer of the decimal fractions.

²That is, if d represents the diameter of a circle, then $3d + \frac{10}{71}d < \text{circumference} < 3d + \frac{1}{7}d$, and hence the ratio of the circumference of a circle to its diameter is less than $3\frac{1}{7}$ and larger than $3\frac{10}{71}$. This is the famous Archimedes' bounds for π ; that is, Proposition 3 of his treatise *On the Measurement of the Circle*.

³It was customary to write the integral part of a number in decimal system and the fractional part in sexagesimal system as we see here.

⁴Here by a hair al-Kāshī meant a horse hair.

⁵Some mathematicians suspect that al-Kāshī may have been motivated by Ptolemy's theorem in stating and proving of what we called al-Kāshī's fundamental theorem. This would be an interesting topic for a future investigation.

⁶This can be obtained from the similar triangles BOM and KOL . Or the similar triangles BOI and KOT .

⁷Al-Kāshī has taken the diameter of the Earth as 2485 *farāsikh*. With this assumption the circumference of the Earth would be $2485 \times \pi \approx 7805$ *farāsikh*, where he has rounded it to 8000 *farāsikh*.

⁸We note that 3×2^{28} is written in decimal system. This is one of the exceptions, for al-Kāshī performed his calculations in sexagesimal system.

⁹That is, he needed to continue his calculations up to 18 sexagesimal digits after the decimal point. In fact he must have continued his calculations up to at least 19 digits after the decimal point in the sexagesimal system, for he used the round-off method.

¹⁰*Amāl* (the plural of *'amal*) means "works" or "calculations".

¹¹This number has been recorded incorrectly as 800335168 in both the original manuscript and the manuscript used by Paul Luckey.

¹²*Marātib* is the plural of *martaba*’ which means position, and here of course it means the position of the decimal point. Thus, the *marātib* of all digits will be reduced once if the decimal point is moved one digit to the left.

¹³Using a modern electronic device to calculate 2π in the sexagesimal system [20] the ninth digit after the decimal point will be 49 and the tenth digit will be 55. However, as we mentioned before, Al-Kāshī used the rounded off method. Therefore, the ninth digit after the decimal point will be 50 and hence as he claimed, all 9 digits of 2π are exact.

¹⁴Al-Kāshī used *al-zā’id* and *al-bāqī* fractions in the title of the Seventh Section. However, both Qurbani [12, p. 133] and Luckey [10, p. 85], translating from Arabic, used *al-nāqīs* instead of *al-bāqī*. Rosenfeld and Hogendijk [15, p. 37] have translated from Persian and used *zā’id* and *nāqīs* as “additive” and “subtractive” and stated that they play the role of the modern terms “positive” and “negative”, respectively.

¹⁵That is, the error will be less than $\frac{1}{60^9}$. He used 60^9 in the denominator because he has calculated 2π up to 9 sexagesimal digits after the decimal point.

¹⁶Using a modern electronic device to calculate 2π in decimal system the sixteenth digit after the decimal point will be 4 and the seventeenth digit will be 7. But, since al-Kāshī used the round-off method his sixteenth digit is 5 and not 4.

¹⁷This approximation of 2π in 1424 was a remarkable achievement for that time, and it far surpassed all approximations of π by all previous mathematicians, astronomers, and other scientists throughout the world including Archimedes, Ptolemy, and Muhammad al-Khwarizmi. After 186 years, Ludolf van Ceulen was the first to improve al-Kāshī’s result by finding the value of π correct to 35 decimal places in 1610.

¹⁸That is, principles that are based on Ptolemy’s *Almagest* for calculating the length of the chord of an arc of a circle, or other similar principles used by astronomers.

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مكتبة آستان قدس

ويوم

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ
 الْحَمْدُ لِلَّهِ الْعَالَمِ بِسْمِهِ الْفَطْرُ إِلَى الْخَطِّ الْيَأْرِفُ
 بِمَقْدَارِ كُلِّ الْمَرْكَبِ وَالْقَبْضِ خَالِقِ الْأَرْضِ وَالسَّمَاءِ
 جَاعِلِ الْمَوَدَّةَ فِي الظُّلُمَاتِ وَالصَّلَوَةَ وَالسَّلَامَ عَلَى
 مُحَمَّدٍ الْمُصْطَفَى مَرَكَزِ دَارِ الرِّيَاءِ وَمَحَطِّ الْفُطَارِ
 الْمُدَائِبِ وَالْعَدَالَةِ وَعَلَى آلِهِ الطَّيِّبِينَ وَأَصْحَابِهِ
 الطَّاهِرِينَ أَمَا بَعْدُ فَيَقُولُ اجْجُجْ خَلَقَ اللَّهُ تَعَالَى
 إِلَى عَمْرَانَةَ جَمِشِيدَنْ مَعُودِيْنَ مُحَمَّدِ الطَّيِّبِ الْكَلْبَانِ
 الْمَلُوقِ بِغَضَائِلِ احْسَنْ لِلَّهِ اِحْوَالَهُ اِنْ اَرْمَيْدَسْ اَنْبِتْ
 اِنْ اَلْحَطِّ اَزْدَنْ تَلْثَهْ اَسْأَلُ الْفَطْرَ بِاَقْلٍ مِنْ سَبْعِ قَطْرِهَا
 وَالْكَثْرَ مِنْ عَشْرَةِ اجْزَاءٍ مِنْ اَحَدٍ وَسَبْعِينَ جُرْءًا مِنْ الْفَطْرِ
 فَالْفَاوِتُ بَيْنَ هَذَيْنِ الْمَقْدَارِ بِنُكُونِ جِزْءٍ اَوْ اجْزَاءٍ
 مِنْ اَرْبَعَاءِهِ وَسَبْعَةٍ وَتِسْعِينَ جُرْءًا فِي دَارٍ بِنُكُونِ
 قَطْرِهَا اَرْبَعَاءَهُ وَسَبْعَةٍ وَتِسْعِينَ دِرْعَامًا اَوْ قِسْمًا اَوْ قُرْحًا
 بِنُكُونِ مَقْدَارِ مَقْطَرِهَا مَحْمُولًا اَوْ شُكُوكًا فِي ذِرَاعٍ وَاَحَدٍ
 اَوْ قِسْمِهِ اَوْ فَرْحٍ وَبِنُكُونِ فِي اعْظَمِ دَارٍ يَقَعُ فِي كَرَةِ الْكَلْبِ
 مَحْمُولًا فِي خَمْسَةِ فَرَاحٍ لَانْ قَطْرُهَا عَمَّهْ اَسْأَلُ كَقَدْرِ الْفَطْرِ
 تَقْرِبًا وَفِي مَنْظَرِهِ فَكُلُّ الْبُرُوجِ مَحْمُولًا فِي الْاَرْضِ مَا يَطْلُفُ
 فَرْحٌ وَهَذِهِ الْمَقَادِرُ فَاجْتَمَعَتْ فِي الْمَقَطَاتِ مَلْفٌ بِنُكُونِ
 فِي الْمَسَاجِدِ وَدَكِّ لَانَّهُ اسْتَفْحَجَ مَحْمُولًا فِي سِتِّهِ وَتِسْعِينَ
 ضِلْعًا فِي الدَّيْرَةِ وَهُوَ اَعْلَى مِنْ مَحْمُولِ الْدَارِ لَانْ كَقَدْرِ الْفَطْرِ
 مِنْ

Plate A: First page of al-Risāla al-muḥīṭīyya
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