No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

169. *Proposed by Dorin Marghidanu, Colegiul National "A. I. Cuza", Corabia, Romania.*

Let $0 < a, b, c < 1$. Prove that

$$2^a(b+c)^{1-a} + 2^b(c+a)^{1-b} + 2^c(a+b)^{1-c} < 4(a+b+c).$$

Solution by Tuan Le (student), Fairmont High School, Anaheim, California. Since $0 < a < 1$, applying Bernoulli's inequality, we have

$$\left(\frac{2}{b+c}\right)^a = \left(1 + \frac{1 - \frac{b+c}{2}}{\frac{b+c}{2}}\right)^a \leq 1 + \frac{a(1 - \frac{b+c}{2})}{\frac{b+c}{2}}.$$

Multiplying both sides of this inequality by $b+c$, we obtain

$$2^a(b+c)^{1-a} \leq 2a + b + c - a(b+c).$$

Similarly, we also obtain

$$2^b(a+c)^{1-b} \leq 2b + a + c - b(a+c)$$

$$2^c(a+b)^{1-c} \leq 2c + a + b - c(a+b).$$

Adding these inequalities together and again using the fact that $0 < a, b, c < 1$, we obtain

$$2^a(b+c)^{1-a} + 2^b(a+c)^{1-b} + 2^c(a+b)^{1-c}$$

$$\leq 4(a+b+c) - 2(ab+bc+ac) < 4(a+b+c).$$

Also solved by Mihai Cipu, Institute of Mathematics of the Romanian Academy, Bucharest, Romania; Fen Qin, Bloomsburg University of Pennsylvania, Bloomsburg, Pennsylvania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Paul Deiermann, Southeast Missouri State University, Cape Girardeau, Missouri; and the proposer.

Paul Deiermann and Mihai Cipu noted that if $n \geq 3$ *and* $a_1, a_2, \ldots, a_n \in (0,1)$*, then the same proof shows that*

$$\sum_{j=1}^{n} \left((n-1)^{a_j} \cdot \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} a_k \right)^{1-a_j} \right) < (2n-2) \sum_{k=1}^{n} a_k .$$

170. *Proposed by Don Redmond, Southern Illinois University, Carbondale, Illinois.*

Let a, b, and n be integers with a and n positive. Suppose that

$$a^n + b = p_1 \cdots p_r ,$$

where $r \geq 1$ and the p's are primes. Let $f = (p_1 - 1) \cdots (p_j - 1)m$, where

$1 \leq j \leq r$ and m is a natural number. Show that $a^{n+fk} + b$ is composite for all natural numbers k.

Solution by Panagiotis T. Krasopoulos, Athens, Greece. First, let us observe that the statement of the problem is not correct and an extra condition is necessary in order to be true. Let $a = 1$, n be a positive integer, $b = p - 1$, where p is a prime. Then $a^n + b = p$ and $a^{n+fk} + b = p$ is a prime number and thus the statement is false. Observe also that it is possible to have $a = 1$ and still the statement is true. Let $b = pq - 1$, where p and q are primes. Then $a^{n+fk} + b = pq$ is a composite number.

Next, we will prove the statement with the addition of an extra condition in order to be always true. We have

$$a^{n+fk} + b = a^{n+fk} + p_1 \cdots p_r - a^n = a^n (a^{fk} - 1) + p_1 \cdots p_r . \qquad (1)$$

Here, it should be assured that the above quantity is not equal to a prime from the list p_1, …, p_r and so we add the extra condition $a \geq 2$ or $r \geq 2$.

Now we have

$$a^{fk} - 1 = \left(a^{(p_1-1)\cdots(p_{j-1}-1)mk}\right)^{p_j-1} - 1. \qquad (2)$$

It is now clear that we can use Fermat's Little Theorem and distinguish two cases.

1) If $p_j|a$, then $p_j|a^n$ and from (1) the quantity $a^{n+fk}+b$ is composite.

2) If p_j does not divide a, then p_j does not divide $(a^{(p_1-1)\cdots(p_{j-1}-1)mk})$.

   Thus, from Fermat's Little Theorem and (2), p_j divides $a^{fk}-1$ and

   so from (1) the quantity $a^{n+fk}+b$ is again composite.

The proof is now complete.

Also solved by Joe Flowers, St. Mary's University, San Antonio, Texas; Mihai Cipu, Institute of Mathematics of the Romanian Academy, Bucharest, Romania; Dmitri Skjorshammer (student), Harvey Mudd College, Claremont, California; and the proposer.

171. *Proposed by José Luis Díaz-Barrero, Universidad Politècnica de Cataluña, Barcelona, Spain.*

Let P be a point in the plane of triangle ABC with sides a, b, and c, respectively. Prove that

$$\left(PA^4 + PB^4 + PC^4\right)^3 \geq \frac{a^4b^4c^4}{27}.$$

When does equality occur?

Solution by the proposer. Let $\mathcal{R} = \{O, \overrightarrow{e_1}, \overrightarrow{e_2}\}$ be a system of reference in the plane of the triangle ABC. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $P(x, y)$, respectively. Then,

$$PA^2 + PB^2 + PC^2 = \sum_{k=1}^{3} \left[(x - x_k)^2 + (y - y_k)^2\right]$$

$$= 3\left[x - \frac{x_1 + x_2 + x_3}{3}\right]^2 + 3\left[y - \frac{y_1 + y_2 + y_3}{3}\right]^2$$

$$+ \frac{1}{3}\Big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2$$

$$+ (y_2 - y_3)^2 + (x_3 - x_1)^2 + (y_3 - y_1)^2\Big)$$

$$\geq \frac{1}{3}(a^2 + b^2 + c^2).$$

Equality holds when $x = (x_1 + x_2 + x_3)/3$, $y = (y_1 + y_2 + y_3)/3$. That is, when $P \equiv G$ (center of gravity of $\triangle ABC$).

Since $PA^2 + PB^2 + PC^2 \geq (a^2 + b^2 + c^2)/3$, then

$$\sqrt{\frac{PA^4 + PB^4 + PC^4}{3}} \geq \frac{PA^2 + PB^2 + PC^2}{3} \geq \frac{a^2 + b^2 + c^2}{9} \geq \frac{1}{3}\sqrt[3]{a^2b^2c^2}.$$

Equality holds when $a = b = c$. From the preceding we obtain

$$\left(\frac{PA^4 + PB^4 + PC^4}{3}\right)^3 \geq \frac{a^4b^4c^4}{3^6}$$

from which the statement immediately follows. Equality holds when $P \equiv G$ and $\triangle ABC$ is equilateral.

Also solved by Tuan Le (student) Fairmont High School, Anaheim, California; and Fen Qin, Bloomsburg University of Pennsylvania, Bloomsburg, Pennsylvania.

172. *Proposed by Ovidiu Furdui, Cluj, Romania.* Find all integer solutions to the Diophantine equation

$$x^4 - x^3 + 1 = y^2.$$

Solution by Mihai Cipu, Institute of Mathematics of the Romanian Academy, Bucharest, Romania. The Diophantine equation has only the eight solutions $(x, y) = (0, \pm 1)$, $(1, \pm 1)$, $(2, \pm 3)$, $(-2, \pm 5)$.

It is easy to check that the first four pairs in the above list are indeed solutions, as well as the fact that there are no solutions with $x = -1$. To find the others, we try to place y^2 between the squares of two consecutive integers.

Let (x, y) be a solution of this Diophantine equation with $|x| \geq 2$. We claim that for x even one has

$$\left(x^2 - \frac{x}{2} - 1\right)^2 < y^2 < \left(x^2 - \frac{x}{2}\right)^2.$$

Indeed, the left inequality is equivalent to $7x^2 > 4x$, which is true for x negative or $x > 4/7$. The right inequality is readily brought to the equivalent form $4 \leq x^2$. Here, the equality holds for $x = \pm 2$, which yields four solutions to our Diophantine equation $(x, y) = (2, \pm 3)$, $(-2, \pm 5)$.

For solutions with x odd we show that one has

$$\left(x^2 - \frac{x-1}{2} - 1\right)^2 < y^2 < \left(x^2 - \frac{x-1}{2}\right)^2.$$

The left inequality, being equivalent to $3x^2 - 2x + 3 > 0$, is true for all real x. The right inequality is equivalent to $5x^2 - 2x - 3 > 0$, which holds for all real x with $x < -3/5$ or $x > 1$. Hence, we conclude that there are no solutions (x, y) to the given Diophantine equation with x odd and $|x| > 2$.

Also solved by Tuan Le (student) Fairmont High School, Anaheim, California; Dmitri Skjorshammer (student), Harvey Mudd College, Claremont, California; Dr. Louis Scheinman, Toronto, Ontario, Canada; and the proposer.