THE NULLITY AND RANK OF COMBINATIONS OF TWO OUTER INVERSES OF A GIVEN MATRIX

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ABSTRACT. This paper shows that the nullity and rank of $aP+bQ-cQAP$ is a constant, where P and Q are outer inverses of a given matrix A, $c=a+b$ $(a,b\neq 0)$ or $c\neq a+b$, $a,b,c\in\mathbb{C}$. In addition, the rank of $aP+bQ-cQAP$ is equal to the rank of $P-Q$ if $c=a+b$ and to $P+Q$ if $c\neq a+b$.

1. Introduction

Let A be an $m\times n$ matrix over the field $\mathbb{C}$ of all complex numbers. A matrix X is said to be an outer inverse of A if $XAX=X$, and is often denoted by $X=A^{(2)}$. The collection of all outer inverses of A is denoted by $A\{2\}$. The outer inverses and their applications have been extensively investigated by many authors in the literature [1, 2, 3]. The rank of $P_1\pm P_2$ and combinations of P_1 and P_2 have been studied by the authors in [4], [5], and [6], where P_1 and P_2 are idempotent matrices. Furthermore, Tian has studied the rank of $P\pm Q$ and linear combinations of P and Q, where P and Q are outer inverses of a given matrix A [7, 8]. In this paper, we study the nullity and rank of combinations $aP+bQ-cQAP$, where P and Q are outer inverses of a given matrix A, $a,b\neq 0$. We prove that the nullity and rank of $aP+bQ-cQAP$ is a constant, where P and Q are outer inverses of a given matrix A, $c=a+b$ $(a,b\neq 0)$ or $c\neq a+b$. In addition, we get the rank equality as follows:

$$r(aP+bQ-cQAP)=\begin{cases} r(P-Q), & \text{when } c=a+b\\ r(P+Q), & \text{when } c\neq a+b,\end{cases}$$

(where P and Q are outer inverses of a given matrix A, $a,b,c\in\mathbb{C}$, $a,b\neq 0$). Our result generalizes the results given by J.J. Koliha and V. Rakočević [4] and Kezheng Zuo [9].

Throughout this paper, we use $\mathbb{C}$, $\mathbb{C}^n$, $\mathbb{C}^{m\times n}$ to denote the set of complex numbers, the n-column vector space over $\mathbb{C}$, and the set of $m\times n$ complex matrices, respectively. If $A\in\mathbb{C}^{m\times n}$, we write $\mathcal{N}(A)$ and $\mathcal{R}(A)$ for the nullspace and the range of A. The rank of A, $r(A)$, is the dimension of $\mathcal{R}(A)$, and the nullity of A, $\mathrm{nul}(A)$, is the dimension of $\mathcal{N}(A)$. The symbol I_n is used to denote the $n \times n$ identity matrix.

The following result is obvious and we omit the proof.

Lemma 1.1. *If $A \in \mathbb{C}^{m\times n}$ be given, $P, Q \in A\{2\}$, we define T as the restriction of $(I_n - PA)Q$ to $\mathcal{N}(P)$, that is,*

$$T\colon \mathcal{N}(P) \longrightarrow [(I_n - PA)Q]\mathcal{N}(P), \;\; x \longmapsto Tx = (I_n - PA)Qx.$$

Then

$$\mathcal{N}(T) = \mathcal{N}[(I_n - PA)Q] \cap \mathcal{N}(P), \mathcal{R}(T) = \mathcal{R}[(I_n - PA)Q(I_m - AP)].$$

2. Main results and Proofs

Now we start our observation with the following result.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ be given, $P, Q \in A\{2\}$, $a, b \in \mathbb{C}\backslash\{0\}$, $c \in \mathbb{C}$ and $c \neq a+b$. If T is defined as in the Lemma 1.1, then $\mathcal{N}(aP+bQ-cQAP)$ is isomorphic to $\mathcal{N}(T)$.*

Proof. Let $\mathcal{N} = \mathcal{N}(aP + bQ - cQAP)$ and $k \neq 0$. First we show that

$$\mathcal{N} \cong (I_m - AP)\mathcal{N} \;\text{ and }\; \mathcal{N}(T) \cong (kI_m - AQ)\mathcal{N}(T). \tag{2.1}$$

Let $x \in \mathcal{N}$ and $(I_m - AP)x = 0$, then $x = APx$, $Qx = QAPx$ and $(aP + bQ - cQAP)x = 0$. Therefore, $Qx = (a + b - c)^{-1}QA(aP + bQ - cQAP)x = 0$, $Px = a^{-1}(aP + bQ - cQAP)x = 0$, and then $x = APx = 0$. Hence, $\mathcal{N} \cong (I_m - AP)\mathcal{N}$. If $x \in \mathcal{N}(T)$ and $(kI_m - AQ)x = 0$, then $Px = 0$, $Qx = PAQx$, and $AQx = kx$. Thus, $Qx = P(kx) = 0$, $kx = AQx = 0$, that is $x = 0$. Hence, $kI_m - AQ$ restricted to acting from $\mathcal{N}(T)$ to $(kI_m - AQ)\mathcal{N}(T)$ is an isomorphism. Next we prove that

$$(I_m - AP)\mathcal{N} \subset \mathcal{N}(T) \;\text{ and }\; (kI_m - AQ)\mathcal{N}(T) \subset \mathcal{N} \;\text{ for some }\; k \neq 0. \tag{2.2}$$

If $x \in \mathcal{N}$, then $(aP + bQ - cQAP)x = 0$. After the multiplication by QA from left side and using the fact that $QAQ = Q$($\{2\}$-inverse), we obtain that $Qx = b^{-1}(c - a)QAPx$, $Px = QAPx$, and $(I_n - PA)Q(I_m - AP)x = (I_n-PA)[b^{-1}(c-a)Px-Px] = 0$, that is $(I_m-AP)x \in \mathcal{N}(T)$. This proves the first inclusion in (2.2).

If $x \in \mathcal{N}(T)$ and $k = b^{-1}(a + b - c) \neq 0$, then $Px = 0$, $Qx = PAQx$. Thus, $(aP + bQ - cQAP)(kI_m - AQ)x = (bk - a - b + c)Qx = 0$. That is, $(kI_m - AQ)x \in \mathcal{N}$. This proves the second inclusion in (2.2). The proof is completed by combining (2.1) and (2.2). □

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times n}$ be given, $P, Q \in A\{2\}$, $a, b \in \mathbb{C}\backslash\{0\}$, $c \in \mathbb{C}$, and $c \neq a + b$. If T is defined as in the Lemma 1.1, then the rank of $aP+bQ-cQAP$ is a constant. Moreover, $r(aP+bQ-cQAP) = r(P+Q) = r(P) + r(T) = m - dim[\mathcal{N}((I_n - PA)Q \cap \mathcal{N}(P)]$.*

Proof. By Theorem 2.1 and Lemma 1.1,

$$\begin{aligned} r(aP+bQ-cQAP) &= m-\mathrm{nul}(aP+bQ-cQAP) \\ &= m-\mathrm{nul}(T) \\ &= m-\dim[\mathcal{N}((I_n-PA)Q)\cap\mathcal{N}(P)], \end{aligned}$$

hence, $r(aP+bQ-cQAP)$ is constant. We set $a=b=1$, $c=0$, then this constant is $r(P+Q)$. According to Lemma 1.1 and Theorem 2.1, $r(T)=\mathrm{nul}(P)-\mathrm{nul}(T)=m-r(P)-\mathrm{nul}(T)=r(aP+bQ-cQAP)-r(P)$, which implies $r(aP+bQ-cQAP)=r(P)+r(T)$. □

Corollary 2.3. *Let $A\in\mathbb{C}^{m\times n}$ be given, $P,Q\in A\{2\}$, $a,b\in\mathbb{C}\backslash\{0\}$, $c\in\mathbb{C}$, and $c\neq a+b$, then the rank of $aP+bQ-cPAQ$ is a constant and equals to $r(P+Q)$.*

Proof. By Theorem 2.2, we obtain $r(aP+bQ-cPAQ)=r(Q+P)=r(P+Q)$. □

Corollary 2.4. *Let $A\in\mathbb{C}^{m\times n}$ be given, $P,Q\in A\{2\}$, $a,b\in\mathbb{C}\backslash\{0\}$, $a+b\neq 0$. Then*

(a) $r(P+Q-QAP)=r(P+Q-PAQ)=r(P+Q)$.
(b) $r(aP+bQ)=r(P+Q)$.

Proof.

(a) In Theorem 2.2 and Corollary 2.3, we set $a=b=c=1$, then

$$r(P+Q-QAP)=r(P+Q-PAQ)=r(P+Q).$$

(b) In Theorem 2.2, we set $c=0$, then $r(aP+bQ)=r(P+Q)$. □

It is obvious that if $A=I_n$ in (b) of Corollary 2.4, then $P,Q\in I_n\{2\}$ are idempotent matrices. Thus, we get the result in the Theorem 2.4 in [4].

Theorem 2.5. *Let $A\in\mathbb{C}^{m\times n}$ be given, $P,Q\in A\{2\}$, $a,b\in\mathbb{C}\backslash\{0\}$, $c\in\mathbb{C}$, and $c=a+b$, then*

$$\mathcal{N}(aP+bQ-cQAP)=\mathcal{N}(aP+bQ-cPAQ)=\mathcal{N}(P-Q).$$

Proof. If $x\in\mathcal{N}(P-Q)$, then $Px=Qx$ and $(aP+bQ-cQAP)x=(a+b-c)x=0$. That is to say $\mathcal{N}(P-Q)\subset\mathcal{N}(aP+bQ-cQAP)$. On the other hand, if $x\in\mathcal{N}(aP+bQ-cQAP)$, then $(aP+bQ-cQAP)x=0$, i.e., $Qx=QAPx=Px$ and $(P-Q)x=0$. Therefore, $\mathcal{N}(aP+bQ-cQAP)\subset\mathcal{N}(P-Q)$. Hence, we obtain the equality $\mathcal{N}(aP+bQ-cQAP)=\mathcal{N}(P-Q)$. Since $\mathcal{N}(P-Q)=\mathcal{N}(Q-P)$, we have finally proved the desired result. □

Remark 2.6. *It is obvious that if $A=I_n$ in Theorem 2.1, Corollary 2.3, and Theorem 2.5, then $P,Q\in I_n\{2\}$ are idempotent matrices which yield the results of* [9].

References

[1] A. Ben-Isrel and T. N. G. Greville, *Generalized inverses: theory and applications*, Springer, 2003, second edition.

[2] Y. Chen and X. Chen, *Representation and approximation of the outer inverse $A^{(2)}_{T,S}$ of a matrix A*, Linear Algebra Appl., **308** (2000), 85–107.

[3] E. P. Liski and S. Wang, *On the {2}-inverse and some ordering properties of nonnegative definite matrices*, Acta Math. Appl. Sinica (English Series), **12** (1996), 22–27.

[4] J. J. Koliha and V. Rakočević, *The nullity and rank of linear combinations of idempotent matrices*, Linear Algebra Appl., **418** (2006), 11–14.

[5] J. Groβ and G. Trenkler, *Nonsingularity of the difference of two oblique projectors*, SIAM J. Matrix Anal. Appl., **21** (1999), 390–395.

[6] Y. Tian and G. P. H. Styan, *Rank equalities for idempotent and involutary matrices*, Linear Algebra Appl., **335** (2001), 101–117.

[7] Y. Tian, *Rank equalities related to outer inverses of matrices and applications*, Linear Algebra Appl., **388** (2004), 279–288.

[8] Y. Tian, *More on rank equalities for outer inverses of matrices with applications*, International Journal of Mathematics, Game Theory and Algebra, **12** (2002), 137–151.

[9] Zuo Kezhing, *The nullity and rank of combinations of idempotent matrices*, Journal of Math., **28** (2008), 619–622.

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