UTILIZING THE EXPANSION OF $P^n - Q^n$ TO INTRODUCE AND DEVELOP THE EXPONENTIAL FUNCTION

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Recently, Bayne et al. [1, 2], have applied the identity

$$P^n - Q^n = (P - Q)\sum_{k=0}^{n-1} P^k Q^{n-1-k} \tag{1}$$

for real P, Q and positive integers n to present simple proofs of the existence of nth roots and inequalities used in real analysis. In this article the identity (1) is used to prove that f defined by

$$f(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

is a real-valued continuous function onto the positive reals with the collection of reals as its domain, and to establish some properties of f, including $f(x+y) = f(x)f(y)$, $f(0) = 1$ and an elegant proof that $f' = f$ where f' represents the derivative function for f. The equation $f(r) = (f(1))^r$ is shown to hold for rational r. This motivates the notation $f(x) = (f(1))^x = e^x$ and calling f the exponential function.

As in [4], the exponential function is often introduced as the inverse of the logarithmic function which is defined as

$$\int_1^x \frac{1}{t}dt.$$

Later, when convergence of sequences is studied, e^x is proved to be the limit of the sequence $(1 + \frac{x}{n})^n$. There again the logarithmic function is used. Dieudonné [3] introduced the logarithmic function by proving that

For any $a > 1$, there is a unique increasing continuous function g of the positive reals into the reals such that $g(xy) = g(x) + g(y)$ and $g(a) = 1$.

The approach adopted here leads to a new proof of the result of Dieudonné and will serve as an exercise on sequences for Calculus students, encouraging them to search for different viewpoints on well-established results.

In what follows, it will be shown that, for each $x \geq 0$, the sequence $(1+\frac{x}{n})^n$ is monotonic and bounded and, hence, converges. The existence of the limit is extended to all real numbers x, by showing that

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = \frac{1}{\lim_{n\to\infty}(1+\frac{x}{n})^n}.$$

Theorem 1. For each real number x, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ exists.

Proof. First it will be established that $\left(1+\frac{x}{n}\right)^n$ is nondecreasing and bounded above for each nonnegative x. The proof is similar to the proof in [1] that $\left(1+\frac{1}{n}\right)^n$ is increasing and bounded above.

Proof that $\left(1+\frac{x}{n}\right)^n$ is nondecreasing. Let $x \geq 0$ and $a_n = \left(1+\frac{x}{n}\right)^n$. Then

$$\begin{aligned} a_{n+1} - a_n &= \left(1+\frac{x}{n+1}\right)^{n+1} - \left(1+\frac{x}{n}\right)^n \\ &= \left(1+\frac{x}{n+1}\right)^{n+1} - \left(1+\frac{x}{n}\right)^{n+1} + \left(1+\frac{x}{n}\right)^{n+1} - \left(1+\frac{x}{n}\right)^n. \end{aligned}$$

It is seen from (1) that

$$\left(1+\frac{x}{n+1}\right)^{n+1} - \left(1+\frac{x}{n}\right)^{n+1} = \frac{-x}{n(n+1)}\sum_{k=0}^{n}\left(1+\frac{x}{n+1}\right)^k\left(1+\frac{x}{n}\right)^{n-k}$$

$$\geq \frac{-x}{n(n+1)}\sum_{k=0}^{n}\left(1+\frac{x}{n}\right)^n = \frac{-x}{n(n+1)}(n+1)\left(1+\frac{x}{n}\right)^n = \frac{-x}{n}\left(1+\frac{x}{n}\right)^n,$$

and clearly,

$$\left(1+\frac{x}{n}\right)^{n+1}-\left(1+\frac{x}{n}\right)^{n}=\left(1+\frac{x}{n}\right)^{n}\left(1+\frac{x}{n}-1\right)=\left(1+\frac{x}{n}\right)^{n}\frac{x}{n}.$$

Therefore,

$$a_{n+1}-a_n\geq\frac{-x}{n}\left(1+\frac{x}{n}\right)^{n}+\frac{x}{n}\left(1+\frac{x}{n}\right)^{n}=0.$$

The proof shows that for positive x, the sequence a_n is strictly increasing.

Proof that a_n is bounded. Consider the difference $\left(1+\frac{x}{mn}\right)^n-1$, where n and m are positive integers with $m>x$. From (1)

$$\left(1+\frac{x}{mn}\right)^{n}-1=\frac{x}{mn}\sum_{k=0}^{n-1}\left(1+\frac{x}{mn}\right)^{k}\leq\frac{x}{mn}\sum_{k=0}^{n-1}\left(1+\frac{x}{mn}\right)^{n}$$

$$=\frac{x}{mn}n\left(1+\frac{x}{mn}\right)^{n}=\frac{x}{m}\left(1+\frac{x}{mn}\right)^{n}.$$

Thus, for positive integers n,

$$\left(1+\frac{x}{mn}\right)^{n}-\frac{x}{m}\left(1+\frac{x}{mn}\right)^{n}=\left(1+\frac{x}{mn}\right)^{n}\left(1-\frac{x}{m}\right)\leq 1.$$

Hence, $\left(1+\frac{x}{mn}\right)^{mn}\left(1-\frac{x}{m}\right)^{m}\leq 1$. Since $\left(1+\frac{x}{n}\right)^{n}$ is nondecreasing and $mn\geq n$, we have

$$\left(1+\frac{x}{n}\right)^{n}\left(1-\frac{x}{m}\right)^{m}\leq 1\quad\text{and so}\quad\left(1+\frac{x}{n}\right)^{n}\leq\left(\frac{m}{m-x}\right)^{m}.$$

Therefore, a_n is bounded. Theorem 1 is completed by employing (1) to show the following.

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\left(1-\frac{x}{n}\right)^n = 1$.

$$0 \le 1 - \left(1 - \frac{x^2}{n^2}\right)^n = \frac{x^2}{n^2}\sum_{k=0}^{n-1}\left(1-\frac{x^2}{n^2}\right)^k \le \frac{x^2}{n^2}\sum_{k=0}^{n-1} 1 = \frac{x^2}{n} \to 0.$$

From Theorem 1 it follows that f defined by $f(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ is a real-valued function with the collection of reals as its domain. In the sequel, f will be this function. In Theorem 2 the identity in (1) is applied to produce some properties of f, including an elegant proof that $f' = f$ where f' represents the derivative function for f.

Theorem 2. The function f is a strictly increasing, continuous, differentiable function onto the positive reals satisfying

(i) $f(x+y) = f(x)f(y)$,
(ii) $f(rx) = (f(x))^r$ for each rational r, and
(iii) $f' = f$.

Proof that f is continuous. For real numbers x and a satisfying $|x-a| < 1$,

$$|f(x) - f(a)| \le \lim_{n\to\infty}\frac{|x-a|}{n}\sum_{k=0}^{n-1}\left(1+\frac{|x|}{n}\right)^k\left(1+\frac{|a|}{n}\right)^{n-1-k} \le |x-a|f(1+|a|).$$

Therefore, $\lim_{x\to a} f(x) = f(a)$.

Proof that $f' = f$. For any $x, a,\ x \ne a$,

$$\frac{\left(1+\frac{x}{n}\right)^n - \left(1+\frac{a}{n}\right)^n}{x-a} = \frac{1}{n}\sum_{k=0}^{n-1}\left(1+\frac{x}{n}\right)^k\left(1+\frac{a}{n}\right)^{n-1-k}.$$

So for any nonnegative $x, a,\ x \ne a$

$$\left(1+\frac{\min\{x,a\}}{n}\right)^{n-1} < \frac{\left(1+\frac{x}{n}\right)^n - \left(1+\frac{a}{n}\right)^n}{x-a} < \left(1+\frac{\max\{x,a\}}{n}\right)^{n-1}.$$

Letting $n \to \infty$,

$$f(\min\{x,a\}) \le \frac{f(x)-f(a)}{x-a} \le f(\max\{x,a\}). \qquad (*)$$

For any nonpositive $x, a,\ x \ne a$

$$\frac{f(x)-f(a)}{x-a} = \frac{f(-x)-f(-a)}{f(-x)f(-a)(-x-(-a))}$$

and from inequality (*)

$$\frac{f(\min\{-x,-a\})}{f(-x)f(-a)} \le \frac{f(-x)-f(-a)}{f(-x)f(-a)(-x-(-a))} \le \frac{f(\max\{-x,-a\})}{f(-x)f(-a)}$$

$$\frac{f(\min\{-x,-a\})}{f(-x)f(-a)} \le \frac{f(x)-f(a)}{x-a} \le \frac{f(\max\{-x,-a\})}{f(-x)f(-a)}. \qquad (**)$$

It follows from inequalities (*), (**), continuity of the functions $f, \max, \min$, and the "squeezing principle" that

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f(a).$$

Proof of (i). For real numbers x and y

$$|f(x+y)-f(x)f(y)| \le \lim_{n\to\infty} \frac{|xy|}{n^2} \sum_{k=0}^{n-1} \left(1+\frac{|x+y|}{n}\right)^k \left(1+\frac{|x+y|}{n}+\frac{|xy|}{n^2}\right)^{n-1-k}$$

$$\le \lim_{n\to\infty} \frac{|xy|}{n} f(|x+y|+|xy|) = 0.$$

Proof of (ii). From (i) and induction, it follows that $f(mx) = (f(x))^m$ for all nonnegative integers m and real x. The identity $f(x)f(-x) = 1$ may then be used to prove $f(mx) = (f(x))^m$ for all integers m and real x. It is now clear that

$$f(1) = f\left(n\left(\frac{1}{n}\right)\right) = \left(f\left(\frac{1}{n}\right)\right)^n \text{ and } (f(1))^{\frac{1}{n}} = f\left(\frac{1}{n}\right)$$

for each positive integer n and hence, $f(r) = (f(1))^r$ for each rational r.

The function f is onto the set of positive reals. Let $z > 0$ and choose an integer m such that $m > z + \frac{1}{z}$. Since $f(1) > 2$, it follows that $f(m) = (f(1))^m > 2^m > m > z + \frac{1}{z} > z$ and that

$$f(-m) = \frac{1}{f(m)} < \frac{1}{z + \frac{1}{z}} < z.$$

By the Intermediate Value Theorem there is an x such that $f(x) = z$.

The function f is strictly increasing. This is a consequence of the facts that $f = f'$ and that f has positive values. It is instructive to see a proof using (1). If x and y are nonnegative and $x < y$ then

$$\begin{aligned} f(y) - f(x) &= \lim_{n\to\infty} \left[\left(1 + \frac{y}{n}\right)^n - \left(1 + \frac{x}{n}\right)^n\right] \\ &= \lim_{n\to\infty} \frac{y-x}{n} \sum_{k=1}^{n-1} \left(1 + \frac{y}{n}\right)^k \left(1 + \frac{x}{n}\right)^{n-1-k} \geq (y-x)f(x) > 0. \end{aligned}$$

If x and y are nonpositive and $x < y$ then $-x$ and $-y$ are nonnegative and $-y < -x$. Hence, $f(-y) < f(-x)$ and $f(x)f(y)f(-y) < f(-x)f(x)f(y)$, so $f(x) < f(y)$. Finally, if $x < y$ and $0 \in [x, y]$, then $f(y) - f(x) = (f(y) - f(0)) + (f(0) - f(x)) > 0$.

Theorem 3. If g is a continuous real-valued function on the reals satisfying

(i) $g(x + y) = g(x)g(y)$ and
(ii) $g(1) = f(1)$,

then $g = f$.

Proof. It will be sufficient to show that $g(r) = f(r)$ for rational r. From (i) and (ii), $f(1) = g(1) = g(1+0) = g(0)g(1) = g(0)f(1)$, so $g(0) = 1$. Hence, $g(x)g(-x) = g(0) = 1$. These properties of g and arguments like those in the proof of Theorem 2 (ii) will establish that $g(r) = (g(1))^r = (f(1))^r = f(r)$ for rational r.

Remark 1. The function f^{-1} is a continuous strictly increasing function from the positive reals onto the reals satisfying $f^{-1}(xy) = f^{-1}(x) + f^{-1}(y)$, $f^{-1}(1) = 0$, and $(f^{-1})'(x) = 1/x$. The function f^{-1} is of course customarily called the logarithm function.

Remark 2. For $a > 0$ and real x, a^x may now be defined as $f(xf^{-1}(a))$.

The final results in this article illustrate an interesting method of proof. Another property of f is offered in Theorem 4.

Theorem 4. For any $z > 0$, some integer m satisfies $f(m) \leq z < f(m+1)$.

Proof. From above, there is an integer n such that $f(n) \leq z$. Let $\mathcal{A}$ be the collection of such $f(n)$ and let $p = \sup \mathcal{A}$. Since $f(1) > 1$ it follows that $p/f(1) < p$. Choose an integer m satisfying $f(m) \leq p$, $p/f(1) < f(m)$, and consequently $p < f(m+1)$. Since $m+1$ is an integer and $f(m+1) \notin \mathcal{A}$, m satisfies $f(m) \leq z < f(m+1)$.

Remark 3. It is interesting that an argument similar to that used in the proof of Theorem 4 produces the following simple proof that between any two distinct reals x and y there is a rational, although it is not as geometrical in nature as the usual proof. (The essence of the technique usually employed is to show that there is an interval $I = [a, b]$ with integer endpoints such that $x,\ y \in I$ and then to partition such an interval I into n subintervals of equal length, where $|x - y| > (b-a)/n$). Suppose $x < y$, Q is the set of rationals, and let $S = \{r \in Q : r < y\}$. Then $S \neq \emptyset$ (the set of integers has no lower bound) and y is an upper bound for S. If $s = \sup S$ then for each positive integer m there is an $r_m \in S$ satisfying $s < r_m + 1/m$. Then $r_m + 1/m \notin S$, $r_m + 1/m \in Q$ and consequently, for such m,

$$\begin{cases} r_m \leq s < r_m + 1/m \\ r_m < y \leq r_m + 1/m. \end{cases} \tag{$***$}$$

From $(***)$, $0 \leq y - s < 1/m$ for each positive integer m and hence, $s = y$. Since $x < y$ there is an $r \in S$ such that $x < r$.

References

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