COVERS IN THE LATTICE OF FUZZY TOPOLOGIES – I

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Abstract. In this paper we combine the concept of simple extensions of a fuzzy topology with closure structure of fuzzy points to give certain necessary and sufficient conditions for a fuzzy topology to have a cover in the lattice of fuzzy topologies.

1. Introduction. The concept of covers in the lattice of topologies and consequently that of adjacent topologies was introduced by Pushpa Agashe and Norman Levine in [1]. In this paper we study the concept of covers in the lattice of fuzzy topologies and consequently that of adjacent fuzzy topologies. In [3] the authors investigated certain classes of fuzzy points based on the structure of their closure. In this note we establish how these classes of fuzzy points together with the idea of simple extensions give rise to certain necessary and sufficient conditions for a fuzzy topology to have a cover.

2. Preliminaries. In this section, we include certain definitions needed for the subsequent development. We denote by I_d, the discrete fuzzy topology in the lattice τ_d of all fuzzy topologies on a set X.

Definition 2.1. Let (X, F) be an fts. A fuzzy set g in X is said to be generalized closed (g-closed) if and only if cl $g \leq f$ whenever $g \leq f$ and $f \in F$ (cl g denotes the smallest closed fuzzy set containing g).

Definition 2.2. Let (X, F) be an fts. A fuzzy set g in X is said to be nearly crisp if cl $g \wedge \mathbb{C}(\text{cl } g) = \underline{0}$.

Definition 2.3. Let (X, F) be an fts. A fuzzy point x_λ in X is said to be well-closed if cl x_λ is again a fuzzy point in X.

Definition 2.4. Let (X, F) be an fts. Then it is called a c-fts if the members of F are characteristic functions.

3. Simple Extensions and Covers.

Definitions 3.1 [3]. Let (X, F) be an fts and suppose that $g \in I^X$ and $g \notin F$. Then the collection $F(g) = \{g_1 \vee (g_2 \wedge g) \; ; \; g_1, g_2 \in F\}$ is called the simple extension of F determined by g.

Theorem 3.2 [3]. Let (X, F) be an fts and suppose that $F(g)$ is the simple extension of F determined by g. Then
(i) $F(g)$ is an fts on X and
(ii) $F(g) = F \vee G$ where $G = \{\underline{0}, \underline{1}, g\}$.

Definition 3.3. Let F and G be two fuzzy topologies on a set X. Then F imp G if and only if (i) $F < G$ and (ii) if $F \leq H \leq G$ where H is an fts on X, then either $F = H$ or $H = G$. We call F and G adjacent if and only if F imp G or G imp F. Also if F imp G then we say that F is an immediate predecessor of G and G is an immediate successor of F or G is a cover of F.

Theorem 3.4 [3]. Let F and G be two fuzzy topologies on a set X such that F imp G. Then G is a simple extension of F.

Theorem 3.5. I_d is not a cover of any fuzzy topology on a set X.

Proof. To prove this theorem it suffices to show that for any $g \notin F$, $F(g) \neq I_d$. Given any $g \notin F$, we can find fuzzy subsets $f_1, f_k, g_1, g_k \neq g$ such that $f_1 \vee f_k = g = g_1 \wedge g_k$ with $f_1, g_1 \notin F$. Now repeating this argument on g with f_1, g_1 and subsequent fuzzy subsets we get a sequence of fuzzy subsets as follows:

$$\cdots \leq f_2 \leq f_1 \leq g \leq g_1 \leq g_2 \leq \cdots$$

where $f_i, g_i \neq g$ and $f_i, g_i \notin F$, for every i. Also by the denseness property of real numbers, it follows that the choice of f_i and g_i for any i is uncountable. Hence, we must have $F(g) \neq I_d$.

As a consequence of the above theorem, we get the following result in [2].

Corollary 3.6. There is no dual atom in τ_d.

In [3], the following theorem was proved.

Theorem 3.7. Let (X, F) be a c-fts and let x_γ; $\gamma = 1$ be a closed fuzzy point in X. Then for any fuzzy point x_λ, $F(\mathbb{C}(x_\lambda))$ is a cover of F.

In the above theorem, F was restricted to a c-fts. But in the following theorem we remove that restriction and prove the result for any arbitrary fts.

Theorem 3.8. Let (X, F) be any fts in which the fuzzy point x_γ; $\gamma = 1$ is closed. Then for any fuzzy point x_λ, $F(\mathbb{C}(x_\lambda))$ is a cover of F.

Proof. Let U be any fts on X such that $F \leq U \leq F(\mathbb{C}(x_\lambda))$; $F \neq U$. Consider any $f \in F(\mathbb{C}(x_\lambda))$ such that $f \in U - F$. Then $f = f_1 \vee (f_2 \wedge \mathbb{C}(x_\lambda))$; $f_1, f_2 \in F$.

Since x_γ; $\gamma = 1$ is closed, we have $f \vee \mathbb{C}(x_\gamma) = \mathbb{C}(x_\lambda) \in U$. Then by Theorem 3.2 we get $U = F(\mathbb{C}(x_\lambda))$ and hence, $F(\mathbb{C}(x_\lambda))$ is a cover of F.

The following theorem gives a necessary condition for a fuzzy topology to have a cover.

Theorem 3.9. Let (X, F) be an fts such that every $g \notin F$ contains a g-closed and nearly crisp fuzzy point x_λ; $\lambda \neq 1$ with $g(x) = \lambda$. If for each such g, there exists an $h \in F$ containing x_λ with $h \vee g \notin F$, then F has no covers.

Proof. To prove that F has no covers, we will show that for any $g \notin F$, the simple extension $F(g)$ is not a cover of F. Since $g \notin F$, there exists a g-closed and nearly crisp fuzzy point x_λ; $\lambda \neq 1$ with $g(x) = \lambda$. Also, we can find an $h \in F$ containing x_λ such that $g \vee h \notin F$. Then we must have $F < F(g \vee h) \leq F(g)$.

If possible, let $g \in F(g \vee h)$. Then $g = f_1 \vee (f_2 \wedge (g \vee h))$; $f_1, f_2 \in F$. Then either $x_\lambda \in f_1$ or $x_\lambda \notin f_1$. Since x_λ is g-closed and nearly crisp, in both cases we get $g(x) = 1$, a contradiction. Thus, we get $F < F(g \vee h) < F(g)$. Then by Theorem 3.4, F has no covers.

The following theorem gives a sufficient condition for a fuzzy topology to have a cover.

Theorem 3.10. Let (X, F) be an fts. If it contains a well-closed but not closed fuzzy point, then F has a cover.

Proof. Let x_λ be a well-closed but not closed fuzzy point in (X, F). We claim that $F(\mathbb{C}(x_\lambda))$ is a cover of F. Suppose that $F \leq G \leq F(\mathbb{C}(x_\lambda))$; $F \neq G$. Let $g \in G - F$. Then $g = g_1 \vee (g_2 \wedge \mathbb{C}(x_\lambda))$; $g_1, g_2 \in F$. Define $h = \mathbb{C}(\text{cl } x_\lambda) \in G$ so that $h \vee g = \mathbb{C}(x_\lambda) \in G$. Then by Theorem 3.2, $G = F(\mathbb{C}(x_\lambda))$ and hence, $F(\mathbb{C}(x_\lambda))$ is a cover of F.

References

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