A CHARACTERIZATION OF PARACOMPACTNESS IN TERMS OF FILTERBASES

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In elementary courses in general topology, students study filterbases and encounter characterizations of compact spaces, Lindelöf spaces, countably compact spaces, etc. in terms of filterbases. Later, when studying paracompact spaces, students ask for a characterization of such spaces in terms of filterbases. The purpose of this note is to make available such a characterization. We assume familiarity with filter subbases and filterbases from N. Bourbaki [1] and all spaces are topological spaces. Recall that a collection of subsets of a space is locally finite if each point in the space has a neighborhood which has nonempty intersection with at most finitely many members of the collection. A family of subsets Ω refines a family of subsets Γ if each $A \in \Omega$ satisfies $A \subset B$ for some $B \in \Gamma$. A space X is paracompact if for each covering Λ of X by open sets, there is a covering of X by open sets which is locally finite and which refines Λ [2].

Definition 1. In a space X a collection of subsets Ω of X is *locally ultimately dominating (l.u.d.)* if for each $x \in X$ there is an open set about x contained in all but finitely many elements of Ω.

Definition 2. A filterbase Ω on a space is of type $\mathcal{P}$ if each l.u.d. filter subbase coarser than Ω has nonempty adherence (we say that a filter subbase Ω is coarser than a filterbase Γ if each element of Ω contains an element of Γ).

We offer the following characterization of paracompact spaces. Although, customarily, the Hausdorff separation axiom is assumed in the definition of paracompact spaces, no separation axioms are assumed in this note. If Ω is a filter subbase on a space the adherence of Ω will be denoted by adh Ω.

Theorem 1. A space is paracompact if and only if every filterbase of type $\mathcal{P}$ on the space has nonempty adherence.

Before giving the proof of Theorem 1 we state Lemma 1 without proof.

Lemma 1. A filterbase is of type $\mathcal{P}$ if and only if every coarser l.u.d. closed filter subbase has nonempty adherence.

Proof of Theorem 1. For the necessity part of the proof, let the space X be paracompact and let Ω be a filterbase on X such that adh $\Omega = \emptyset$. Then $\{X - \overline{F} : F \in \Omega\}$ is an open cover of X which has a locally finite open refinement κ. Then

$\kappa^* = \{X - G : G \in \kappa\}$ is a filter subbase on Xcoarser than Ω, κ^* is l.u.d. and adh $\kappa^* = \emptyset$. Hence, Ω is not of type $\mathcal{P}$. For the sufficiency, suppose each filterbase of type $\mathcal{P}$ on X has nonempty adherence, and let Ω be an open cover of X with no finite subcover. Then $\{X - \bigcup_\Gamma A : \Gamma \subset \Omega,\ \Gamma \text{ finite }\}$ is a filterbase on X with empty adherence which, from Lemma 1, has a coarser l.u.d. closed filter subbase Λ with empty adherence; then $\{X - F : F \in \Lambda\}$ is locally finite. For each $F \in \Lambda$ choose a finite $\Omega(F) \subset \Omega$ such that $X - \bigcup_{\Omega(F)} A \subset F$, so $X - F \subset \bigcup_{\Omega(F)} A$. For $F \in \Lambda$, let $\mathcal{H}(F) = \{A \cap (X - F) : A \in \Omega(F)\}$ and let $\mathcal{R} = \bigcup_{F\in\Lambda} \mathcal{H}(F)$. Clearly $\mathcal{R}$ is an open refinement of the original open covering. Also

$$\bigcup_{\mathcal{R}} V = \bigcup_{F\in\Lambda} [(X - F) \cap \bigcup_{\Omega(F)} A] = \bigcup_{F\in\Lambda} (X - F) = X.$$

We show that $\mathcal{R}$ is locally finite. If $x \in X$ there is an open set V about x with $V \cap (X - F) = \emptyset$ for all but finitely many $F \in \Lambda$. Let Σ be the finite subset of Λ such that $F \in \Sigma$ implies $V \cap (X - F) \neq \emptyset$. If $Q \in \mathcal{R}$ and $Q \cap V \neq \emptyset$, it follows that $Q = A \cap (X - F) \in \mathcal{H}(F)$, where $F \in \Sigma$ and hence, V has nonempty intersection with at most finitely many elements of $\mathcal{R}$. The proof is complete.

The characterization of paracompactness in Theorem 1 may be utilized in conjunction with the relationship between continuity of filterbases of type $\mathcal{P}$ established in Lemma 2 below to shorten proofs of known results involving continuity and paracompactness. Some examples are provided in the form of Theorems 2, 3, and 4, and Corollary 1.

Lemma 2. If $f\colon X \to Y$ is continuous and Ω is a filterbase of type $\mathcal{P}$ on X, then $\{f(F) : F \in \Omega\}$ is a filterbase of type $\mathcal{P}$ on Y.

Proof. Let Γ be a closed l.u.d. subbase on Y coarser than $\{f(F) : F \in \Omega\}$. Then $\{f^{-1}(G) : G \in \Gamma\}$ is easily shown to be a closed l.u.d. subbase coarser than Ω. Hence, $\bigcap_{G\in\Gamma} f^{-1}(G) \neq \emptyset$. Since $f(\bigcap_{G\in\Gamma} f^{-1}(G)) \subset \bigcap_{G\in\Gamma} G$, it follows that adh $\Gamma \neq \emptyset$. This completes the proof.

Lemma 3 follows easily from Lemma 2.

Lemma 3. If X is a space and $A \subset B \subset X$ then any filterbase of type $\mathcal{P}$ on A is a filterbase of type $\mathcal{P}$ on B.

Proof. The identity function from A to B is continuous. An application of Lemma 2 completes the proof.

Theorem 2. A closed subspace of a paracompact space is paracompact.

Proof. By Lemma 3, a filterbase of type $\mathcal{P}$ on $A \subset X$ is a filterbase of type $\mathcal{P}$ on X and thus has nonempty adherence in X. If A is closed then such filterbases have nonempty adherence in A. The proof is complete.

Theorem 3. Each subspace of a paracompact space is paracompact if and only if each open subspace is paracompact.

Proof. The necessity is obvious. Now suppose A is a subspace of X and that Ω is a filterbase of type $\mathcal{P}$ on A such that $A \cap \text{adh } \Omega = \emptyset$. Then $A \subset X - \text{adh } \Omega$, so Ω is a filterbase on $X - \text{adh } \Omega$ of type $\mathcal{P}$. This is a contradiction since $(X - \text{adh } \Omega) \cap \text{adh } \Omega = \emptyset$. The proof is finished.

Theorem 4. If Y is paracompact and $f: X \to Y$ is a continuous closed function such that $f^{-1}(v)$ is compact for each $v \in Y$ then X is paracompact.

Proof. It will be enough to establish that each closed filterbase on X of type $\mathcal{P}$ has nonempty adherence. If Ω is such a filterbase then $\{f(F) : F \in \Omega\}$ is a closed filterbase on Y of type $\mathcal{P}$ so $\bigcap_\Omega f(F) \neq \emptyset$. For $v \in \bigcap_\Omega f(F)$, $\{F \cap f^{-1}(v) : F \in \Omega\}$ is a closed filterbase on the compact set $f^{-1}(v)$. Hence, $\text{adh } \Omega \neq \emptyset$. The proof is complete.

Corollary 1. If X is compact and Y is paracompact, the product space $X \times Y$ is paracompact.

Proof. The projection $p: X \times Y \to Y$ is continuous, closed, and $p^{-1}(v)$ is homeomorphic to X. Hence, Theorem 4 applies and the proof is complete.

Theorems 2, 3, and 4 and Corollary 1 are due to Dieudonné [2].

References

1. N. Bourbaki, *General Topology*, Hermann, Paris, France; Addison Wesley, Redding, Massachusetts, 1966.

2. J. Dieudonné, "Une Généralisation des Espaces Compacts," *J. Math. Pures Appl.*, 23 (1944), 65–76.

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