THE DIMENSION OF INTERSECTION OF $k$ SUBSPACES

Yongge Tian

Abstract. This note presents a formula for expressing the dimension of intersection of $k$ subspaces in an $n$-dimensional vector space over an arbitrary field $\mathcal{F}$.

It is a well-known fundamental formula in linear algebra that for any two subspaces $V_1$ and $V_2$ in an $m$-dimensional vector space $V$, the dimension of the intersection $V_1 \cap V_2$ is

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2).$$

Written in matrix form, it is equivalent to

$$\dim[R(A_1) \cap R(A_2)] = \rk(A_1) + \rk(A_2) - \rk[A_1, A_2],$$

where $A_1 \in \mathcal{F}^{m \times n_1}$, $A_2 \in \mathcal{F}^{m \times n_2}$, $R(\cdot)$ stands for the range (column space) of a matrix.

In this note we extend it to the $k$-term case to establish a formula for calculating the dimension of the intersection of $k$ column spaces $R(A_1), R(A_2), \ldots, R(A_k)$ using ranks and generalized inverses of matrices.

Let $A$ be an $m \times n$ matrix over an arbitrary field $\mathcal{F}$. A generalized inverse of $A$ is defined as a solution of the matrix equation $AXA = A$ and often denoted by $A^{-}$. For the homogeneous matrix equation $AX = 0$, where $X \in \mathcal{F}^{n \times p}$, its general solution can be written as $X = (I_n - A^{-}A)U$, where $U \in \mathcal{F}^{n \times p}$ is arbitrary [2].

Two basic rank equalities used in the sequel related to generalized inverses [1] are

$$\rk[A, B] = \rk(A) + \rk(B - AA^{-}B),$$

and

$$\rk\begin{bmatrix} A \\ C \end{bmatrix} = \rk(A) + \rk(C - CA^{-}A).$$

Our main result is given below.
Theorem. Let \( A_i \in \mathbb{F}^{m \times n_i}, i = 1, 2, \ldots, k \). Then the dimension of the intersection \( \bigcap_{i=1}^{k} R(A_i) \) is

\[
\dim \left( \bigcap_{i=1}^{k} R(A_i) \right) = \text{rk}(A_1) + \text{rk}(A_2) + \cdots + \text{rk}(A_k) - \text{rk} \left[ \begin{array}{ccc}
A_1 & A_2 & A_3 \\
\vdots & \ddots & \vdots \\
A_1 & & A_k
\end{array} \right].
\]

(5)

Proof. In order to prove (5), we first find a general expression of vectors in \( \bigcap_{i=1}^{k} R(A_i) \). Let \( X \in \bigcap_{i=1}^{k} R(A_i) \). Then it is obvious that there must exist \( X_i \in \mathbb{F}^{n_i \times 1} \) such that

\[
X = A_1X_1 = A_2X_2 = \cdots = A_kX_k.
\]

(6)

Consider it as a system of linear matrix equations, we can write it as

\[
\begin{bmatrix}
I_m & -A_1 \\
I_m & -A_2 \\
\vdots & \ddots \\
I_m & -A_k
\end{bmatrix}
\begin{bmatrix}
X \\
X_1 \\
\vdots \\
X_k
\end{bmatrix} = 0,
\]

(7)

or simply \( MY = 0 \). Solving for \( Y \), we then get \( Y = (I_t - M - M^t)U \), where \( U \in \mathbb{F}^{t \times 1} \), \( t = m + n_1 + \cdots + n_k \). In that case, the general expression of \( X \) is

\[
X = [I_m, 0, \cdots, 0]Y = [I_m, 0, \cdots, 0](I_t - M - M^t)U.
\]
Thus the dimension of $\cap_{i=1}^{k} R(A_i)$, by (4) is

$$\dim\left( \bigcap_{i=1}^{k} R(A_i) \right) = \operatorname{rk}[I_m, 0, \ldots, 0](I_t - M^{-1}M)$$

$$= \operatorname{rk}\begin{bmatrix} I_m & 0 & \cdots & 0 \\ I_m & -A_1 \\ \vdots & \vdots & \ddots & \vdots \\ I_m & -A_k \end{bmatrix} - \operatorname{rk}\begin{bmatrix} I_m & -A_1 \\ \vdots & \vdots \\ I_m & -A_k \end{bmatrix}$$

$$= m + \operatorname{rk}\begin{bmatrix} -A_1 \\ \vdots \\ -A_k \end{bmatrix} - m - \operatorname{rk}\begin{bmatrix} A_1 & -A_2 \\ A_1 & -A_3 \\ \vdots & \vdots \\ A_1 & -A_k \end{bmatrix}$$

$$= \operatorname{rk}(A_1) + \operatorname{rk}(A_2) + \cdots + \operatorname{rk}(A_k) - \operatorname{rk}\begin{bmatrix} A_1 & A_2 \\ A_1 & A_3 \\ \vdots & \vdots \\ A_1 & A_k \end{bmatrix},$$

establishing (5).

**Corollary.** Let $A_i \in F^{m \times n_i}, \ i = 1, 2, \ldots, k$. Then $\cap_{i=1}^{k} R(A_i) = \{0\}$ holds if and only if

$$\operatorname{rk}\begin{bmatrix} A_1 & A_2 \\ A_1 & A_3 \\ \vdots & \vdots \\ A_1 & A_k \end{bmatrix} = \operatorname{rk}(A_1) + \operatorname{rk}(A_2) + \cdots + \operatorname{rk}(A_k).$$

(8)
References


Yongge Tian
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario, Canada K7L 3N6
email: ytian@mast.queensu.ca