AN APPLICATION OF SB-RINGS

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Abstract. All rings are commutative rings with identity and $J(R)$ denotes the Jacobson radical of a ring R. A ring R is called a SB-ring provided that for any sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in R with $s \geq 2$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$, there exists $b \in R$ such that $(a_1, a_2, \ldots a_s, a_{s+1}) = (a_1, a_2, \ldots, a_s + ba_{s+1})$. By applying some of the properties of SB-rings, it is shown that $R[X]$ is not a Prüfer domain for any Noetherian domain R which is not a field.

Preliminaries and the Main Result. All rings are commutative rings with identity and $J(R)$ denotes the Jacobson radical of a ring R. For any $s \geq 1$, a sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in a ring R is called a unimodular sequence provided that $(a_1, a_2, \ldots, a_s, a_{s+1}) = R$. R is said to be a B-ring, if for any unimodular sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in R with $s \geq 2$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$, there exists $b \in R$ such that $(a_1, a_2, \ldots, a_s + ba_{s+1}) = R$. R is said to be a strongly B-ring (SB-ring) provided that for any sequence $a_1, a_2, \ldots, a_s, a_{s+1}$ of elements in R with $s \geq 2$ and $(a_1, a_2, \ldots, a_{s-1}) \not\subseteq J(R)$, there exists $b \in R$ such that $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1, a_2, \ldots, a_s + ba_{s+1})$. For a detailed study of B-rings and SB-rings, see [2]. Furthermore, for a more general case of B-type rings see the dissertation of the author [3].

A Prüfer domain is an integral domain in which every nonzero finitely generated ideal is invertible. A Dedekind domain is an integral domain in which every nonzero ideal is invertible.

Lemma 1. If R is a Dedekind domain, then R is a SB-ring.

Proof. See Theorem 3.2 in [2].

Lemma 2. $R[X]$ is a SB-ring if and only if R is a field.

Proof. See Theorem 3.4 in [2].

Theorem. If R is a Noetherian domain which is not a field, then $R[X]$ cannot be a Prüfer domain.

Proof. Suppose $R[X]$ is a Prüfer domain. Since every ideal in a Noetherian domain is a finitely generated ideal, then $R[X]$ must be a Dedekind domain. Now by applying Lemma 1 and Lemma 2 above, we can conclude that R is a field and this is a contradiction to the choice of R.

Remark. From the above theorem, it is easy to see that $Z[X]$ is not a Prüfer domain, where Z is the ring of rational integers. See also [1] for an argument that shows $Z[X]$ is not a Prüfer domain.

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References

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