GENERALIZED INVOLUTIONS ON BANACH SPACES

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Abstract. We introduce generalized involutive operators on Banach spaces, based on the notion of a spectral family.

1. Introduction. In [7], among others, involutive operators of arbitrary order m, on a Banach space X, are examined (i.e. operators $T: X \to X$, such that $T^m = I$ for a positive integer m); it is proved that $T = \sum_{j=1}^{m} z_j P_j$, where $\{z_j\}$ is the set of the m roots of 1, and P_j are projections on X such that $P_k P_j = 0$ for $k \neq j$ and $\sum_{j=1}^{m} P_j = I$. Thus, it is concluded that involutive operators are scalar on X (i.e. there exists a spectral measure $\mu(\cdot)$ on X such that $T = \int_{\mathbb{R}} \lambda d\mu(\lambda)$).

The above result can be derived immediately as an application of Proposition 10.6 in [4], can serve as a motivation to examine structural features of a slightly more general version of these operators, namely when $\{z_j\}$ is an arbitrary finite subset of the unit circle. For evident reasons we call these (scalar) operators generalized involutive (g.i.) on X. There are two ways of spectrally classifying these operators; by use of a spectral measure or through the concept of a spectral family of projections. We shall follow the second approach. In Section 2 we characterize these operators as trigonometrically well-bounded deriving their spectral decomposition. (All pertinent theoretical tools are presented later on.) Similar results appear in Section 3 by constructing a normalized logarithm for g.i. operators, and by explicitly calculating also the corresponding "ergodic" operators following the formulas of the abstract setting of [2].

We present now, in brief, an account of the basic concepts and results needed throughout. By the term "operator on X" we will always refer to a bounded linear transformation with domain a (complex) Banach space X and range in X. We denote by $B(X)$ the algebra of all operators on X and the prefix "s" denotes the strong topology of $B(X)$. We take for granted the basic theory concerning spectral measures on Banach spaces. Supplementary theory concerning our machinery can also be found in [1,2,4].

Definition 1.1. A spectral family in X is a uniformly bounded projection valued-function $E(\cdot): \mathbb{R} \to B(X)$ which is s-right continuous on $\mathbb{R}$, has an s-left limit on $\mathbb{R}$ and satisfies: (i) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$ for all λ, μ in $\mathbb{R}$ with $\lambda \leq \mu$ (ii) $s\text{-}\lim E(\lambda) = I$ (resp. 0) as $\lambda \to +\infty$ (resp. $\to -\infty$).

If there is a compact $[a,b] \subset \mathbb{R}$ such that $E(\lambda) = 0$ for $\lambda < a$, $E(\lambda) = I$ for $\lambda \geq b$, we say that $E(\cdot)$ is concentrated on $[a,b]$. A theory of integration (in the Riemann-Stieltjes sense) with respect to spectral families is described in full detail in [4]. In particular for $J = [a,b]$ and f in $BV(J)$ we can define $\int_J f dE$ as an s-limit of sums, where the intermediate point in each partitioning subinterval is taken to be the right-end point of the subinterval. If, in addition, f is continuous on J, we can use arbitrary intermediate points [4].

The next definition is an equivalent characterization for a class of operators in $B(X)$, that does not involve (directly) the original description (via the functional calculus introduced in [5] and [6]); it makes use of the integral previously defined. Let us introduce the symbol $\int_J^{\oplus} f dE$ to describe the expression $f(a)E(a) + \int_{[a,b]} f dE$ [1].

Definition 1.2. Let T in $B(X)$. T shall be called well-bounded of type (B), if there exists a compact interval $J \subset R$ and a spectral family $E(\cdot)$ concentrated on J such that

$$T = \int_J^{\oplus} \lambda dE(\lambda).$$

(In this case $E(\cdot)$ is unique and is called the spectral family of T.)

We next define a class of operators that lies in the core of our considerations. We use again an equivalent reformulation of the original definition, convenient for our purposes.

Definition 1.3. An operator T in $B(X)$ shall be called trigonometrically well-bounded if it can be expressed as

$$T = \int_{[0,2\pi]}^{\oplus} \exp(i\lambda) dE(\lambda),$$

where $J = [0, 2\pi]$ and $E(\cdot)$ a spectral family concentrated on J.

Remark. We can always arrange, in Definition 1.3, $E(2\pi^-) = I$ and then $E(\cdot)$ is uniquely determined and shall be simply called the spectral decomposition of T.

The next "multi-theorem" provides a convenient characterization for trigonometrically well-bounded operators, as well as the ground for constructing their "normalized" logarithm and the spectral families in both cases; it is a blending of results in [1] and [2] and it completes the main bulk of the theory needed throughout.

Theorem 1.4. Let T be in $B(X)$. Then

(i) T is trigonometrically well-bounded if and only if $T = \exp(iS)$, for a well-bounded operator S of type (B) on X.

(ii) If T is trigonometrically well-bounded, then there is a (unique) well-bounded operator A of type (B) on X, such that $T = e^{iA}$, $\sigma(A) \subset [0, 2\pi]$ and 2π is not in the point spectrum of A. (Notation $A = \text{Arg}\, T$, argument of T.)

(iii) If T and A are as in (ii), then the spectral decomposition of T coincides with the spectral family of A.

(iv) If T is as in (ii) and it is also "power-bounded" on a reflexive Banach space X, i.e. $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$, then the argument of T is given by

$$\text{Arg}\, T = \pi I - \pi Q_0 + iB_0,$$

where

$$Q_0 = s - \lim_n \frac{1}{n} \sum_{k=0}^{n-1} T^k \quad \text{and} \quad B_0 = s - \lim_n {\sum_{k=-n}^{n}}' \frac{T^k}{k}$$

exist in $B(X)$ (and the "superscript prime" for the second series indicates omission of $n = 0$).

(v) If T is trigonometrically well-bounded and power-bounded on a reflexive Banach space X, then the spectral decomposition of T is given by

$$E(\lambda) = \frac{1}{2\pi i}\{i\lambda I - B_\lambda + B_0\} + \frac{1}{2}\{Q_\lambda + Q_0\} \quad \text{for} \quad 0 \le \lambda < 2\pi,$$

where

$$Q_\lambda = s - \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \exp(-ik\lambda) T^k$$

and

$$B_\lambda = s - \lim_n {\sum_{k=-n}^{n}}' \exp(-ik\lambda) \frac{T^k}{k}$$

are (automatically) in $B(X)$.

Remarks. 1) In Theorem 1.4(v) the reflexivity of X is used in order to ensure that Q_λ is in $B(X)$ (by use of the theory of V.III.5 in [3]); since for g.i. operators we can easily check – which actually we will see later – that this is true on any Banach space, we have not required reflexivity (see Lemma 3.1). 2) In case X is a Hilbert space, we see that g.i. operators on X are simply unitary on X (the corresponding projections taken as self-adjoint). This is not accidental. In fact, any power-bounded operator on a Hilbert space is similar to a unitary operator on the same space, and certainly our operators are power-bounded since

$$T^k = \sum_{j=1}^{m} z_j^k P_j$$

and thus,

$$\sup\{\|T^k\| : k \in \mathbb{Z}\} \leq \sum_{j=1}^{m} \|P_j\| < \infty.$$

(*Scholium:* The above mentioned remarkable feature of power bounded operators on a Hilbert space – which is a consequence of deep theorems due to *Sz*-Nagy has also a rather elementary proof (e.g. see J.A. Van Casteren's "Generators of Strongly Continuous Semigroups," *Research Notes in Mathematics*, No. 115, Pitman, 1985)).

2. G.I. Operators as Trigonometrically Well-Bounded Operators. Throughout this section let

$$T = \sum_{j=1}^{m} z_j P_j,$$

for an arbitrary but fixed positive integer m; to avoid unnecessary fiddling, we shall assume $T \neq I$ (i.e. we shall avoid the case $m = 1$, $z_1 = 1$).

Note that if $T = I$ its spectral decomposition is 0 for $\lambda < 0$ and I for $\lambda \geq 0$.

Now let θ_j be the principal argument of z_j in $[0, 2\pi)$. Based on the theory of integration with respect to a spectral family is not hard to guess that the natural candidate for the spectral decomposition of T is given by

$$E(\lambda) = \sum_{\theta_j \leq \lambda}^{m} P_j, \quad \text{for each } \lambda \in R, \tag{$*$}$$

where the lower index symbolism indicates summation with respect to all j, $1 \leq j \leq m$, such that $\theta_j \leq \lambda$ (with the convention $\sum_{\emptyset} = 0$).

Remark. By definition, it is evident that $E(\lambda) = 0$ for $\lambda < 0$ and $E(\lambda) = I$ for $\lambda \geq 2\pi$.

The two lemmas which follow will demonstrate that the above claim for the spectral decomposition of T is true. We shall denote convergence with respect to the uniform topology of $B(X)$ by "u".

Lemma 2.1. The function $E(\cdot)$ defined on $\mathbb{R}$ via $(*)$ is a spectral family (concentrated on $[0, 2\pi]$).

Proof. Evidently, each $E(\lambda)$ is a projection on X, and since

$$\sup_{\lambda} \|E(\lambda)\| \leq \sum_{j=1}^{m} \|P_j\| < \infty,$$

they form a uniformly bounded family on $\mathbb{R}$. Property (i) of Definition 1.2 is immediate by the properties of the P_j's and (ii) is a direct consequence of the preceding remark (and it actually holds in the u-sense). We now need only to demonstrate s-right continuity and existence of the s-left hand limit of $E(\lambda)$ at each λ in $\mathbb{R}$. For a simpler notation we do that by showing, respectively,

$$u - \lim_{\epsilon \to 0^+} E(\lambda + \epsilon) = E(\lambda) \tag{a}$$

and

$$u - \lim_{\epsilon \to 0^+} E(\lambda - \epsilon) = \sum_{\theta_j < \lambda} P_j. \tag{b}$$

For (a) it is enough to observe that if $\epsilon > 0$, then

$$E(\lambda+\epsilon) - E(\lambda) = \sum_{\lambda<\theta_j\leq\lambda+\epsilon} P_j \underset{\epsilon\to 0^+}{\longrightarrow} \sum_{\emptyset} = 0.$$

Similarly working for (b), we have

$$E(\lambda-\epsilon) = E(\lambda) - \sum_{\lambda-\epsilon<\theta_j\leq\lambda} P_j \underset{\epsilon\to 0^+}{\longrightarrow} E(\lambda) - \sum_{\theta_j=\lambda} P_j = \sum_{\theta_j<\lambda} P_j.$$

<u>Lemma 2.2</u>. For T as before and $E(\cdot)$ defined via $(*)$, we have

$$T = \int_{[0,2\pi]}^{\oplus} \exp(i\lambda) dE(\lambda)$$

(as a strong, and in fact, uniform limit of Riemann-Stieltjes sums).

<u>Proof</u>. It is enough to show that

$$T = u - \lim_{\phi} \left\{ E(0) + \sum_{k=1}^{n} \exp(i\lambda_k)(E(\lambda_k) - E(\lambda_{k-1})) \right\},$$

over all partitions $\phi = \{\lambda_0, \ldots, \lambda_n\}$ of $[0, 2\pi]$.

Note at first that $E(0)$ corresponds to 0 if $z_j \neq 1$ for all j, and to P_{j_0} if $z_{j_0} = 1$ for some (unique) $1 \leq j_0 \leq m$. Observe next that

$$\sum_{k=1}^{n} \exp(i\lambda_k)(E(\lambda_k) - E(\lambda_{k-1})) = \sum_{k=1}^{n} \exp(i\lambda_k) \Bigg(\sum_{\lambda_{k-1}<\theta_j\leq\lambda_k} P_j \Bigg).$$

Since

$$\max_{1\leq k\leq n} |\lambda_k - \lambda_{k-1}|$$

becomes very small as n becomes large, then

$$\sum_{\lambda_{k-1}<\theta_j\leq\lambda_k} P_j$$

is either 0 or P_{j_0} (whenever $\theta_{j_0}=\lambda_k$), and the result follows.

We summarize the results of the two lemmas in the following theorem.

Theorem 2.3. Let

$$T=\sum_{j=1}^{m}\exp(i\theta_j)P_j,$$

for a (fixed) $m\geq 1$, where θ_j are distinct in $[0,2\pi)$ and P_j are projections on a (fixed) Banach space X, such that $P_kP_j=0$ for $k\neq j$ and

$$\sum_{j=1}^{m}P_j=I \quad \text{(the identity on } X).$$

Then (i) T is trigonometrically well-bounded on X, and its spectral decomposition is given by

$$E(\lambda)=\sum_{\theta_j\leq\lambda}P_j, \quad \text{for each } \lambda \text{ in } \mathbb{R}.$$

(ii) T possesses as argument the operator

$$A=\sum_{j=1}^{m}\theta_jP_j \quad \text{whose spectral family is } E(\cdot).$$

Proof. (i) We need only to check that $E(2\pi^-)=I$, and appeal to the remark following Definition 1.3.

(ii)
$$\int_{[0,2\pi]}^{\oplus}\lambda dE(\lambda)=\text{Arg } T$$

by Definition 1.2, Definition 1.3, Theorem 1.4 (ii)–(iii), and the result in (i).

Repeating verbatim the proof of Lemma 2.2, we see that

$$u - \int_{[0,2\pi]}^{\oplus} \lambda dE(\lambda) = \sum_{j=1}^{m} \theta_j P_j.$$

Remark. It is natural at this point to set the question of constructing the spectral decomposition of T (or of the argument of T), without departing from an intuitive observation. First note that for detecting Arg T, an alternative approach could have been the following. Let

$$A = \sum_{j=1}^{m} \theta_j P_j.$$

We easily check that $T = \exp(iA)$ and $\sigma(A) \subset [0, 2\pi]$, since $\sigma(A) = \{\theta_j : 1 \leq j \leq m\}$. To check that A is well-bounded of type (B) without of course involving the previous construction of $E(\lambda)$, we can appeal to the equivalent definition in [1] involving a (weakly compact) $AC([0, 2\pi])$ functional calculus.

The next section reveals how, by rather elementary calculations that involve classical Fourier series we can reproduce, using the formula of Theorem 1.4 (iv)–(v), the results of Section 2.

3. The Construction of the Argument and the Spectral Decomposition of G.I. Operators. Set

$$A = \sum_{j=1}^{m} \theta_j P_j.$$

Since $T = \exp(iA)$ and A is well-bounded of type (B), we conclude that T is trigonometrically well-bounded without appealing to the results of the lemmas in Section 2. Since T is also power-bounded, Theorem 1.4 (iv)–(v) is applicable. We (temporarily) fix λ in $\mathbb{R}$ and proceed to calculate the operators Q_λ, B_λ, in the subsequent lemma. For notational convenience, we set $P_{j,\lambda}$ to be 0 if $\theta_j \neq \lambda$ and P_{j_0} if $\theta_{j_0} = \lambda$, for a (unique) j_0, $(1 \leq j_0 \leq m)$.

We also denote by $E(\lambda)$ the operator

$$\sum_{\theta_j \leq \lambda} P_j.$$

Lemma 3.1. Let T be as in Section 2. Then the operators Q_λ and B_λ defined in Theorem 1.4 (v) have the following representation, for any λ in $\mathbb{R}$.

(i) $Q_\lambda = P_{j,\lambda}$
(ii) $B_\lambda = i\{(\lambda + \pi)I + \pi P_{j,\lambda} - 2\pi E(\lambda) - A\}$.

Proof. (i) It is immediate that

$$Q_\lambda = \sum_{j=1}^{m} \omega_j P_j,$$

where

$$\omega_j = \lim_n \omega_{nj}, \quad \omega_{nj} = \frac{1}{n} \sum_{k=0}^{n-1} \exp(ik(\theta_j - \lambda)).$$

Evidently,

$$\omega_{nj} = \begin{cases} 1, & \text{if } \theta_j = \lambda \\ \frac{1}{n}\{1 - \exp(i(\theta_j - \lambda))\}, & \text{if } \theta_j \neq \lambda, \end{cases}$$

and thus,

$$\omega_{nj} = \begin{cases} 1, & \text{if } \theta_j = \lambda \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Similarly, due to the symmetric summation, we can easily see that

$$B_\lambda = \sum_{j=1}^{m} \delta_j P_j,$$

where

$$\delta_j = s - \lim_n 2i \sum_{k=1}^{n} \frac{\sin k(\theta_j - \lambda)}{k} = 2is - \lim_n \sum_{k=1}^{n} \frac{\sin k(\theta_j - \lambda)}{k}.$$

Based on the classical formula for the Fourier series of

$$\frac{\pi - \theta}{2}$$

for $0 < \theta < 2\pi$, we can easily see that

$$\delta_j = \begin{cases} 0, & \text{for } 0 \le \lambda = \theta_j < 2\pi \\ i(\lambda - \theta_j + \pi), & \text{for } \lambda < \theta_j < 2\pi \\ i(\lambda - \theta_j - \pi), & \text{for } 0 \le \theta_j < \lambda. \end{cases}$$

We conclude that $B_\lambda = \lambda(I - P_{j,\lambda} - (A - \lambda P_{j,\lambda}) - \pi(E(\lambda) - P_{j,\lambda}) + \pi(I - E(\lambda))$, and the result follows.

Theorem 3.2. Let T be an operator as in Theorem 2.3. Then this trigonometrically well-bounded operator has an argument, Arg T, and a spectral decomposition, $F(\cdot)$, such that

(i) Arg $T = A$
(ii) $F(\lambda) = E(\lambda)$ for each λ in $\mathbb{R}$.

Proof.

(i) Theorem 1.4 (iv) implies Arg $T = \pi I - \pi P_{j,0} - (\pi I - \pi P_{j,0} - A) = A$, since $E(0) = Q_0 = P_{j,0}$.

(ii) $F(\lambda) = E(\lambda)$ for $\lambda < 0$ or $\lambda \geq 2\pi$ is immediate, since $F(\cdot)$ is concentrated on $[0, 2\pi]$ and $E(\cdot)$ behaves, by construction, in a similar way. Now let $0 \leq \lambda < 2\pi$. Then by Theorem 1.4 (v),

$$F(\lambda) = \frac{1}{2\pi i}\big\{i\lambda\pi I - B_\lambda + B_0\big\} + \frac{1}{2}\big\{Q_\lambda + Q_0\big\}.$$

Substituting B_λ and B_0 as given in Lemma 3.1 (ii) we can easily obtain the announced equality.

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