RINGS WHOSE MODULES REQUIRE AN INVARIANT NUMBER OF MINIMAL GENERATORS

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Abstract. We examine rings having the property that, for each minimally generated module, the number of elements in a minimal generator set is invariant.

Introduction. Let R be a ring. A left R-module M is a minimally generated module if M contains a subset $S = \{m_i : i \in I\}$ that generates M, and for each i, $m_i \notin \operatorname{span}\{m_j : j \neq i\}$. The set S is a minimal generator set (abbreviated as "mgs") of M. Every left module of a left perfect ring has a minimal generator set. A ring R has the left invariant minimality property (IMP) and is call an IMP ring if, for each minimally generated left R-module M, the number of elements in each minimal generator set of M is invariant. The prefix "left" (IMP) is presumed throughout. Unless stated to the contrary, a module is presumed to be a left module. Every module is presumed to be unitary and every ring has an identity. Throughout, "R" denotes a ring and "J" denotes the Jacobson radical of R.

Theorem 1. A ring R has IMP if and only if R is a local ring.

Proof. Assume R is a local ring with $J = \operatorname{Rad}(R)$. Let $S = \{m_i : i \in I\}$ and $T = \{t_k : k \in K\}$ be minimal generator sets of a minimally generated R-module M. In M/JM we let $S^\wedge = \{m_i + JM : i \in I\}$ and $T^\wedge = \{t_k + JM : k \in K\}$. We will show that the canonical map $p\colon S \to S^\wedge$, where $p(m_i) = m_i + JM$, for all $i \in I$, is a bijection. Suppose that $p(m_k) = p(m_n)$ for some $k \neq n$, then $m_k + JM = m_n + JM$, so $m_k = m_n + \sum r_i m_i$ for some $r_i \in J$, thus $(1 - r_k)m_k = \sum_{i \neq k} r_i m_i + m_n$. Since $r_k \in J = \operatorname{Rad}(R)$, r_k is a unit. $m_k = \sum_{i \neq k} (1 - r_k)^{-1} r_i m_i + (1 - r_k)^{-1} m_n$. This is a contradiction since S is a minimal generator set of M. So p is injective (and clearly surjective). So $\operatorname{card}(S^\wedge) = \operatorname{card}(S)$.

Now we show that $S^\wedge$ is a basis for the vector space M/JM over the division ring R/J. Clearly $S^\wedge$ generates M/JM. Suppose that $S^\wedge$ is not a minimal generator set of M/JM. Then for some k, $m_k + JM = \sum_{i \neq k} (a_i + J)(m_i + JM)$, so $m_k = \sum_{i \neq k} a_i m_i + \sum_i r_i m_i$ for some $r_i \in J$. Hence, $m_k = \sum_{i \neq k} (1 - r_k)^{-1}(a_i + r_i) m_i$, a contradiction since $S = \{m_i : i \in I\}$ is a minimal generator set of M. We have shown that $S^\wedge$ is a basis of M/JM and that $\operatorname{card}(S^\wedge) = \operatorname{card}(S)$. In the same manner it can be shown that the mapping $t_j \to t_j + RM$ is a bijection from T to $T^\wedge$ (so $\operatorname{card}(T^\wedge) = \operatorname{card}(T)$), and that $T^\wedge$ is a basis of M/JM over R/J.

Since any two bases of a vector space have the same cardinality, we know that $\text{card}(S^\wedge) = \text{card}(T^\wedge)$. Thus, $\text{card}(S) = \text{card}(T)$. Hence, the ring R has IMP.

For the converse, assume that R has IMP. To show that R is a local ring we will show that R has only one maximal left ideal. Let M_1 be a maximal left ideal of R. Suppose $x \in R - M_1$. We will show that x is a unit. Since $R = M_1 + Rx$, $1 = r + ax$ for some $r \in M_1$, $a \in R$; so $\{r, x\}$ generates R. Now $\{1\}$ is a minimal generator set of R with one element, and since R has IMP, every mgs of R must have exactly one element. Thus, $\{r\}$ or $\{x\}$ is a mgs of R. Now $\{r\}$ does not generate R. So $\{x\}$ is a mgs of R. Hence, $1 = yx$ for some $y \in R$. Suppose $xy \neq 1$. Then $1 - xy = e$ is a non-trivial idempotent and $R = Re \oplus R(1-e)$. Hence, $\{e, 1-e\}$ is a minimal generator set of R with two elements, a contradiction. Hence, $xy = 1$; so x is a unit. Therefore, R has only one maximal left ideal. This proves that R is a local ring [1].

Definition. Let C be a class of R-modules. A ring R has the invariant minimality property for modules in C if, for each minimally generated R-module in C, the number of elements in a minimal generator set is invariant.

Comments.

(A) The proof of Theorem 1 shows that the following are equivalent:
  (1) Ring R has IMP;
  (2) Ring R has IMP for the class of cyclic R-modules;
  (3) Ring R has IMP for the class of free R-modules;
  (4) Every minimal generator set of the module $_RR$ contains exactly one element.

(B) Ring R has the left IMP if and only if ring R has the right IMP (since both conditions are equivalent to R being a local ring).

(C) The existence of a minimal generator set for each R-module does not imply that the ring R has IMP. Let R be a left perfect ring that is not a local ring. Every left R-module of a left perfect ring has a mgs [5], but R doesn't have IMP.

Corollary 1. A ring R has IMP if and only if the ring R/J has IMP.

Proof. R is a local ring if and only if R/J is a local ring. The statement follows from Theorem 1.

Corollary 2. Let R be a ring such that each left R-module has a minimal generator set. Then, R has IMP if and only if R is a left perfect local ring.

Proof. Assume R has IMP. Then R is a local ring. Let M be a non-zero R-module. The non-zero, minimally generated module M contains a maximal submodule [5], thus, $JM \subseteq \text{Rad}(M) \neq M \Rightarrow M \neq JM$. Hence, $J = \text{Rad}(R)$ is left

T-nilpotent [1]. Since ring R/J is semisimple (being a division ring) and J is left T-nilpotent, R is a left perfect ring [1]. Conversely, if we assume that R is a local ring then R has IMP by Theorem 1.

Corollary 3. Let R be a semisimple ring. Then R has IMP if and only if R is a division ring.

Proof. Assume R is a semisimple ring with IMP. Since R is a semisimple ring, $J = \text{Rad}(R) = 0$. Since R has IMP, R is a local ring; thus, $R/J = R/\{0\}$ is a division ring. Hence, R is a division ring.

For the converse, if we assume that R is a division ring, then R is a local ring, which implies R has IMP.

In [4] a ring is defined to be an invariant basis number (IBN) ring if for every free module, the number of elements in a basis is invariant. It is proven as a consequence of [4] that a local ring is an invariant basis number ring. This result follows directly from Theorem 1.

Corollary 4. If R is a local ring, then R is an invariant basis number ring.

Proof. If R is a local ring then R has IMP by Theorem 1. Since a basis for a free left R-module F is also a minimal generator set of F, it follows that any two bases of F must have the same cardinality. So R is an IBN-ring.

Definition. Let M be an R-module. M is finitely related if there exist an exact sequence $0 \to G \to F \to M \to 0$ of finitely generated R-modules F and G, with F free.

Definition. Let M be a minimally generated R-module and let $S = \{m_i : i \in I\}$ be a mgs of M. Let F be a free R-module with basis $\{f_i : i \in I\}$. Consider the epimorphism $p\colon F \to M$ defined by $\sum a_i f_i \to \sum a_i m_i$ and let $K = \ker(f)$. If for some mgs S of M, K is a minimally generated R-module, then we define M to be a minimally related A-module.

It is known that every finitely related flat module is projective [4]. Over a local ring this extends to a minimally related flat module.

Theorem 2. Every non-zero, flat, minimally related module of a local ring is a free module.

Proof. Let M be a non-zero, flat, minimally related R-module of a local ring R. Let $S = \{m_i : i \in Q\}$ be a mgs of M and let F be a free module with basis $\{f_i : i \in Q\}$ such that the homomorphism $f\colon F \to M$ defined by $\sum a_i f_i \to \sum a_i m_i$ has a minimally generated kernel K. Let $x = \sum a_i f_i \in K$. Since S is a mgs of M, each a_i is a non-unit, and since R is a local ring, each a_i must be in $J = \text{Rad}(R)$, so

K is contained in JF. Since M is a flat module, $K \cap JF = JK$, so $K = JK$. Assume that $K \neq 0$. Because K is minimally generated, K has a maximal submodule, so $K \neq \text{Rad}K$, and thus, $K \neq JK$; a contradiction. Therefore, $K = 0$, so $F \approx M$, and it follows that M is a free module.

References

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